Combinatorial Hopf Algebras.

YORK U UNIVERSITÉ UNIVERSITY

Nantel Bergeron

York Research Chair in Applied Algebra

www.math.yorku.ca/bergeron

[with J.Y. Thibon ... ... and many more]

Outline

- What would be a good gift for a mathematician?
- What is a Combinatorial Hopf Algebra?
- Sym is a strong, realizable CHA with character.
- On strong CHA (categorification)
- On realizable CHA (word combinatorics and quotients).

Combinatorial Hopf Algebra

$H = \bigoplus_{n\geq 0} H_n$ a graded connected Hopf algebra is CHA if

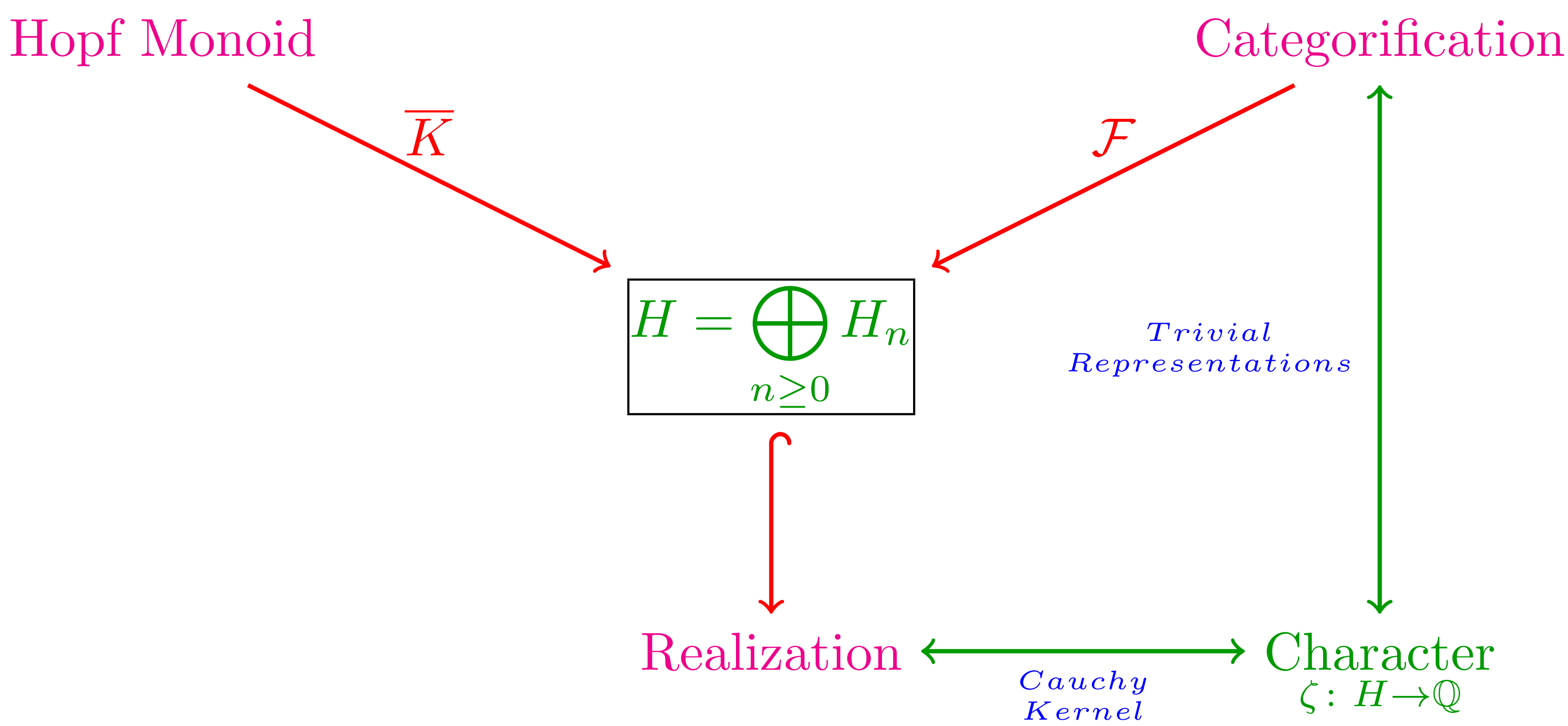
(weak) There is a distinguished (combinatorial) basis with positive integral structure coefficients (from Hopf monoid).

(strong) The structure is obtained from representation operation (from categorification).

(real.) It can be realized in a space of series in variables. (it is realizable)

(char.) It has a distinguished character. (with character)

Combinatorial Hopf Algebra



Sym is the model CHA

Sym is the space of symmetric functions $\mathbb{Z}[h_1, h_2, \ldots]$, with $\deg(h_k) = k$ and

$$\Delta(h_k) = \sum_{i=0}^{k} h_i \otimes h_{k-i}.$$

Sym is the model CHA

$\boxed{Sym}$

Sym is the model CHA

Sym is the space of symmetric functions $\mathbb{Z}[h_1, h_2, \ldots]$, with $\deg(h_k) = k$ and

$$\Delta(h_k) = \sum_{i=0}^{k} h_i \otimes h_{k-i}.$$

It is the functorial image of a Hopf Monoid Π:

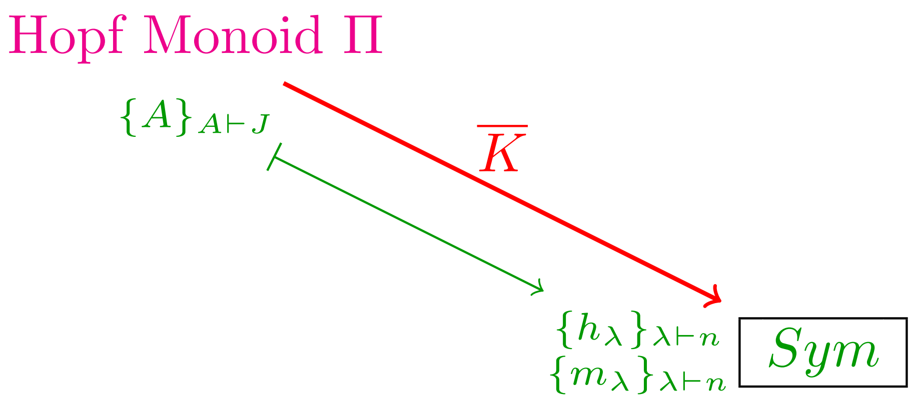
For any finite set J let $\Pi[J] = \{A : A \vdash J\}$ the set partitions of J.

Product and Coproduct:

combinatorial constructions on set partitions

It correspond to flats of the hyperplane arrangement of type A.

Sym is the model CHA



Hopf structure on $\bigoplus_{n\geq 0} K_0(S_n)$

$K_0(S) = \bigoplus_{n\geq 0} K_0(S_n)$ is the space of S_n-modules up to isomorphism

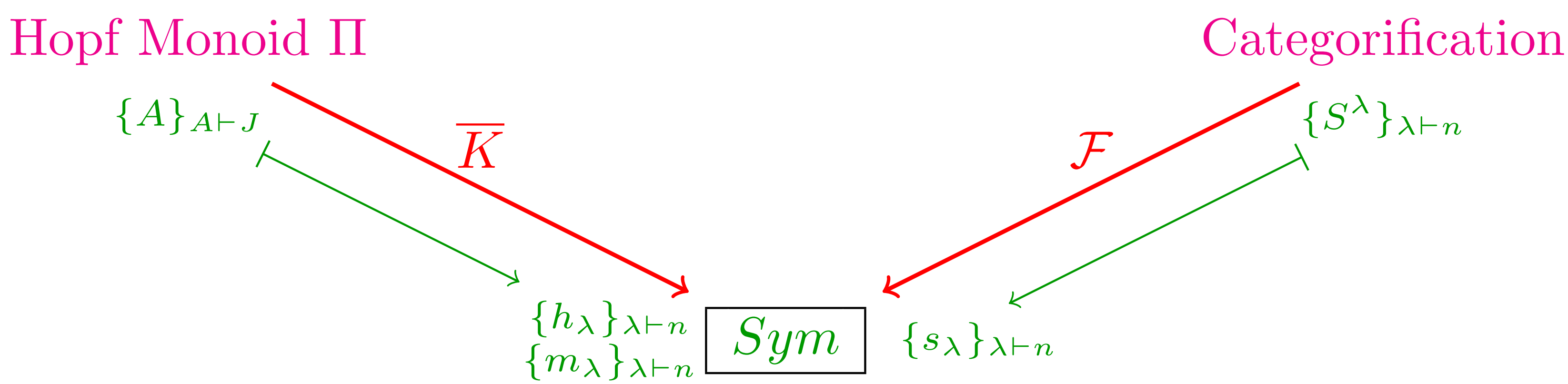
- Basis: Irreducible modules S^λ
- Structure:

$$M * N = \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} M \otimes N$$

$$\Delta M = \bigoplus_{k=0}^{n} \mathrm{Res}_{S_k \times S_{n-k}}^{S_n} M$$

- $\mathcal{F}$: $K_0(S) \to Sym$ is an isomorphism of graded Hopf algebra where $\mathcal{F}(S^\lambda) = s_\lambda$

Sym is the model CHA

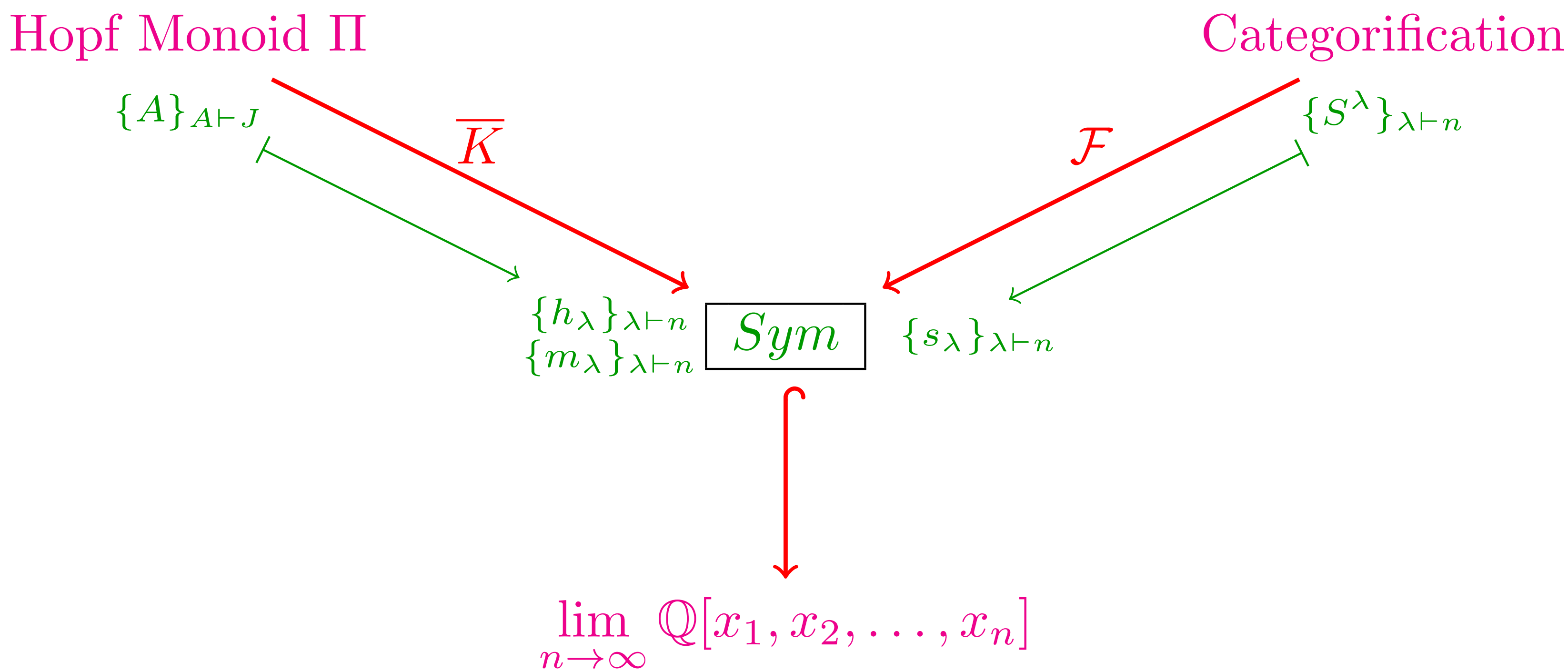


Realization of Sym

$Sym \hookrightarrow \lim_{n\to\infty} \mathbb{Q}[x_1, x_2, \ldots, x_n]$

Allows us to understand coproducts, internal coproduct, plethysm, Cauchy kernel, ...

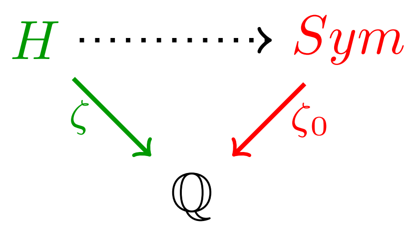
Sym is the model CHA



Sym with a Hopf Character

$$\begin{aligned} \zeta_0 : \quad & Sym & \to \quad & \mathbb{Q} \\ & f(x_1, x_2, \ldots) & \mapsto \quad & f(1, 0, \ldots) \end{aligned}$$

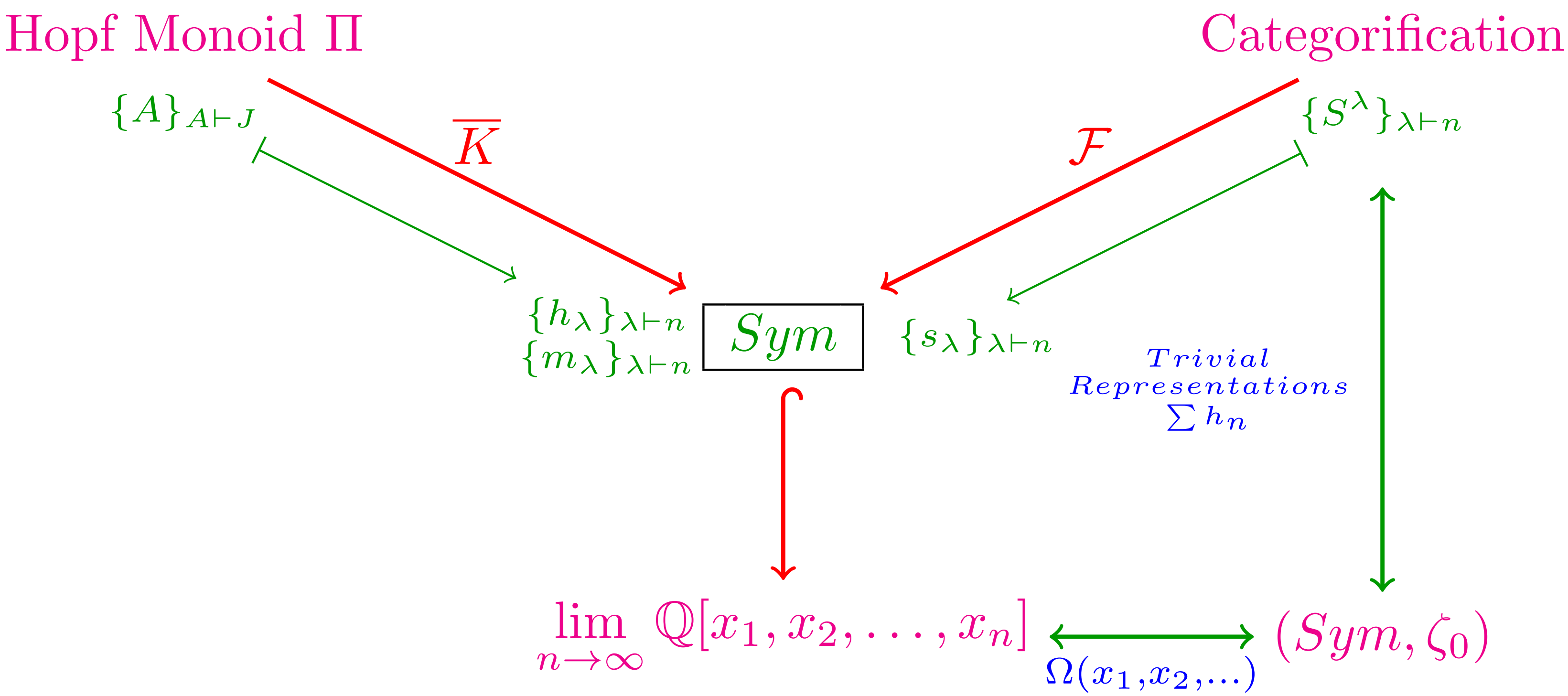
(Sym, ζ_0) is a terminal object for (H, ζ) cocommutative:



$$\zeta_0^* = \sum_{n \geq 0} h_n$$

$$\Omega(X) = \sum_{n \geq 0} h_n(X) = \prod_{x \in X} \frac{1}{1 - x}$$

Sym is the model CHA



Toward Categorification

Consider a graded algebra $A = \bigoplus_{n\geq 0} A_n$

- Each A_n is an algebra.
- $\dim A_0 = 1$ and $\dim A_n < \infty$.
- $\rho_{n,m} : A_n \otimes A_m \hookrightarrow A_{n+m}$; injective algebra homomorphism
- A_{n+m} is projective bilateral submodule of $A_m \otimes A_m$.
- Right and left projective structure of A_{n+m} are compatible.
- There is a Mackey formula linking induction and restriction

A is a tower of algebra

Toward Categorification

Consider a tower of algebras $A = \bigoplus_{n \geq 0} A_n$

Let $K_0(A) = \bigoplus_{n \geq 0} K_0(A_n)$ is the space of (projective) A_n-modules up to isomorphism and modulo short exact sequences

- $K_0(A)$ is a graded Hopf algebra:

$$M * N = \mathrm{Ind}_{A_n \otimes A_m}^{A_{n+m}} M \otimes N$$

$$\Delta M = \bigoplus_{k=0}^{n} \mathrm{Res}_{A_k \otimes A_{n-k}}^{A_n} M$$

- H is a strong CHA if there is an isomorphism

$$\mathcal{F} \colon K_0(A) \to H$$

Example of Tower of Algebras

$\mathbb{Q}S = \bigoplus_{n\geq 0} \mathbb{Q}S_n$:

$$\mathcal{F}\colon K_0(\mathbb{Q}S) \to Sym$$

$H(0) = \bigoplus_{n\geq 0} H_n(0)$: [Krob-Thibon]

$$\mathcal{F}\colon K_0(H(0)) \to NSym$$

$$\mathcal{F}\colon G_0(H(0)) \to QSym$$

$HC(0) = \bigoplus_{n\geq 0} HC_n(0)$: [B-Hivert-Thibon] ... Peak algebras ...

seams rare?

Obstruction to Tower of algebras?

Consider a tower of algebras $A = \bigoplus_{n\geq 0} A_n$

where $K_0(A)$ and $G_0(A)$ are graded dual Hopf algebra:

THEOREM[B-Lam-Li]

> if A is a tower of algebras, then $\dim(A_n) = r^n n!$

this is very restrictive...

Tower of Supercharacters [... B ... Novelli ... Thibon ...]

- Unipotent upper triangular matrices over finite Fields $\mathbf{F}_q$: $U_n(q)$.
- Superclasses in $U_n(q)$: $A \cong B \quad \leftrightarrow \quad (A - I) = M(B - I)N$
- Supercharacters χ: characters constant on superclasses:

$$\Delta(\chi) = \sum_{A+B=[n]} \mathrm{Res}^{U_n(q)}_{U_{|A|}(q)\times U_{|B|}(q)} \chi$$

$$\chi \cdot \psi = \mathrm{Inf}^{U_{n+m}(q)}_{U_n(q)\times U_m(q)} \chi \otimes \psi = (\chi \otimes \psi) \circ \pi$$

where $\pi : U_{n+m}(q) \to U_n(q) \times U_m(q)$.

- $\mathcal{F} : K_0\Big(\bigoplus_{n\geq 0} U_n(2)\Big) \to NCSym$ is iso.

$NCSym$ symmetric functions in non-commutative variables.

Some open questions

(Q-1) Find other examples of Categorification (Can we do $NCQsym$ (quasi-symmetric in non commutative variables)?

(Q-2) Tower of algebra A (axiomatization with superclasses/ supermodules and Harish-Chandra induction:

$$\mathrm{Ind} \circ \mathrm{Inf} \qquad \text{and} \qquad \mathrm{Def} \circ \mathrm{Res} \qquad).$$

About Realization

Many CHA are realized: Sym, NSym , QSym, NCSym, • • •

Can we described all

$$H \hookrightarrow \mathbb{Q}\langle x_1, x_2, \ldots\rangle$$

with monomial basis (equivalence classes on words) [Giraldo].

[B-Hohlweg] Monomial basis embeddings

$$H \hookrightarrow SSym$$

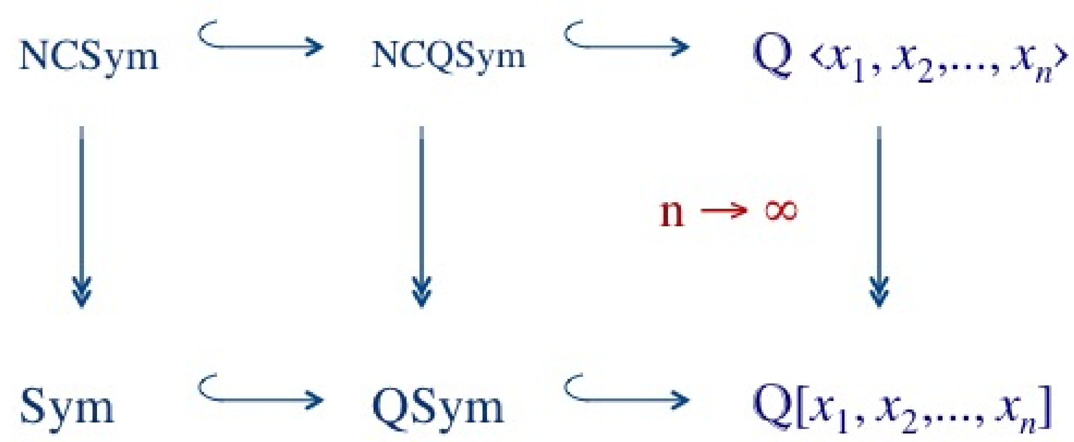
(Q-3) Realization Theory: Can we describe monomial embeddings

$$H \hookrightarrow \mathbb{Q}M$$

for different monoid M

Realization Quotients

Hopf algebras



1

Jan 2005

Reverse Lex and Gröbner basis

$$\mathbb{Q}[x_1,\ldots,x_{n+1}] \xrightarrow{x_n=0} \mathbb{Q}[x_1,\ldots,x_n]$$

$$H[x_1,\ldots,x_{n+1}] \xrightarrow{x_n=0} H[x_1,\ldots,x_n]$$

G_n G-basis of ideal $\langle H[x_1,\ldots,x_n]^+\rangle$:

$$G_{n+1} \xrightarrow{x_n=0} G_n$$

$$g(x_1,\ldots,x_{n+1}) \longmapsto \begin{cases} 0 & \text{if } LT(g)|_{x_n=0}=0 \\ \tilde{g} & \text{if } LT(g)|_{x_n=0}=LT(\tilde{g})\neq 0 \end{cases}$$

B_n basis of quotient $\mathbb{Q}[x_1,\ldots,x_n]\big/\langle H[x_1,\ldots,x_n]^+\rangle$:

$$B_{n+1} \hookleftarrow B_n$$

Reverse Lex and Gröbner basis

$$\mathbb{Q}[x_1,\ldots,x_{n+1}] \xrightarrow{x_n=0} \mathbb{Q}[x_1,\ldots,x_n]$$

$$H[x_1,\ldots,x_{n+1}] \xrightarrow{x_n=0} H[x_1,\ldots,x_n]$$

$$G_{n+1} \xrightarrow{x_n=0} G_n$$

$$g(x_1,\ldots,x_{n+1}) \longmapsto \begin{cases} 0 & \text{if } LT(g)\big|_{x_n=0}=0 \\ \tilde{g} & \text{if } LT(g)\big|_{x_n=0}=LT(\tilde{g})\neq 0 \end{cases}$$

$$B_{n+1} \hookleftarrow B_n$$

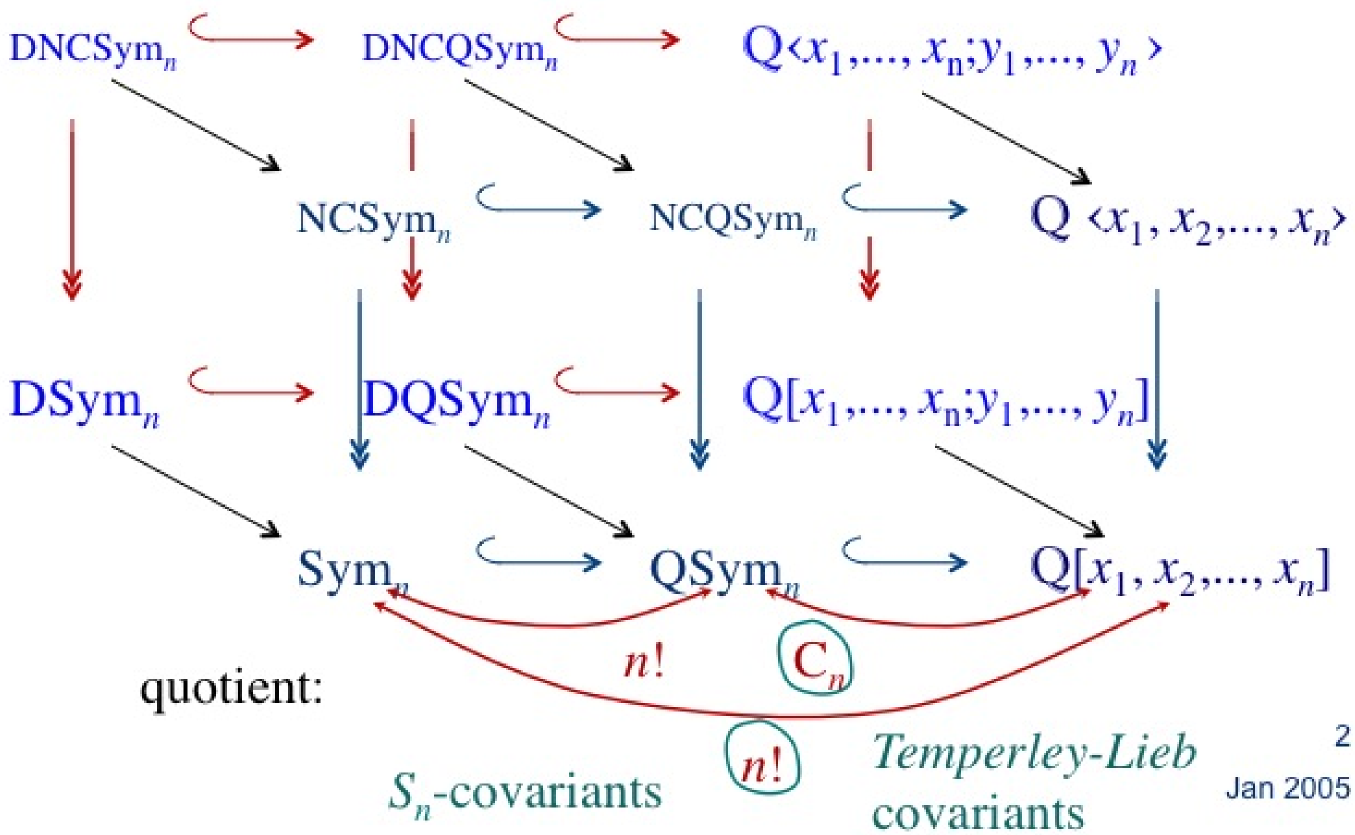
$$\xleftarrow{\textit{mult by } x_n}$$

$$\xleftarrow{\textit{mult by } x_n^2}$$

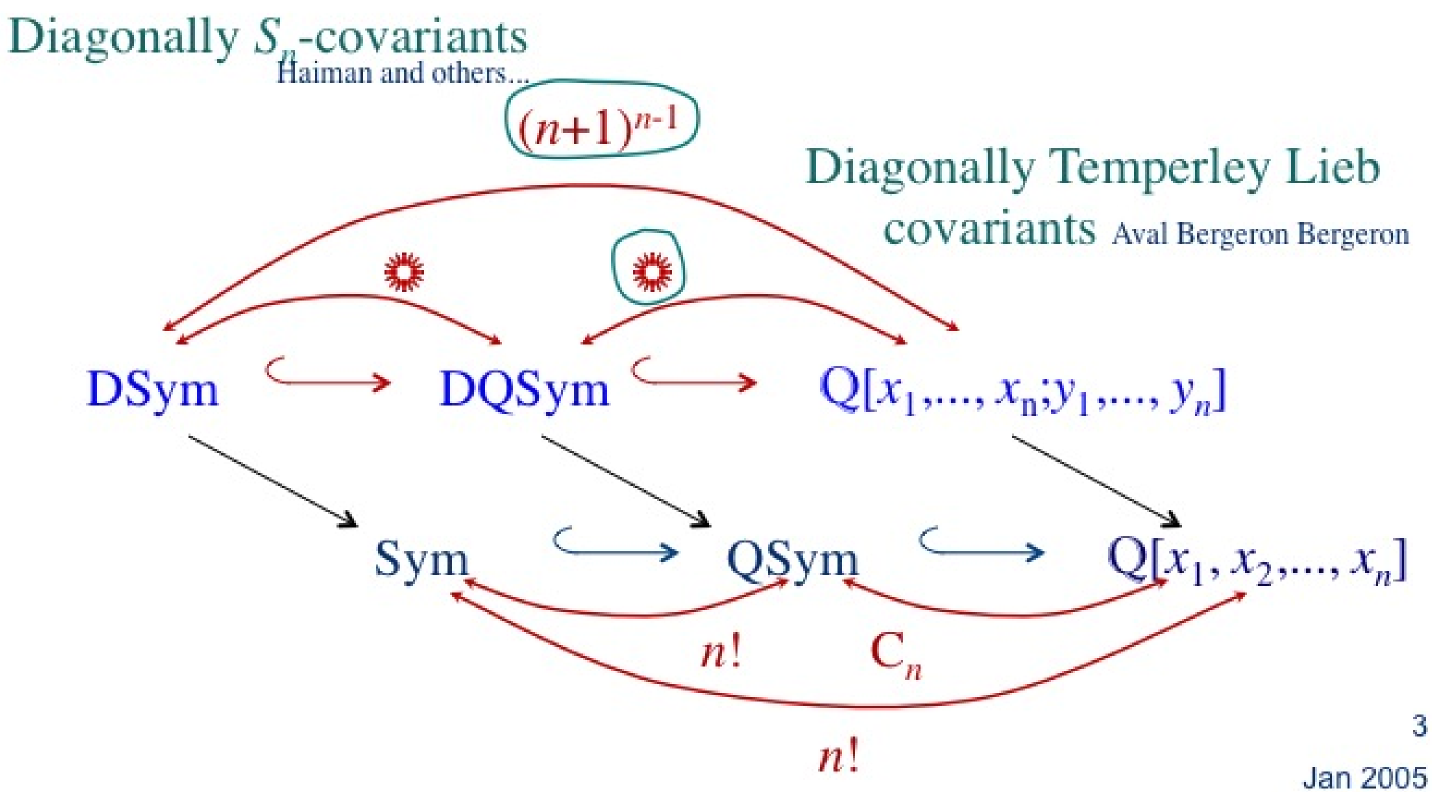
$$\xleftarrow{\textit{mult by } x_n^3}$$

$\cdots$

Realization Quotients



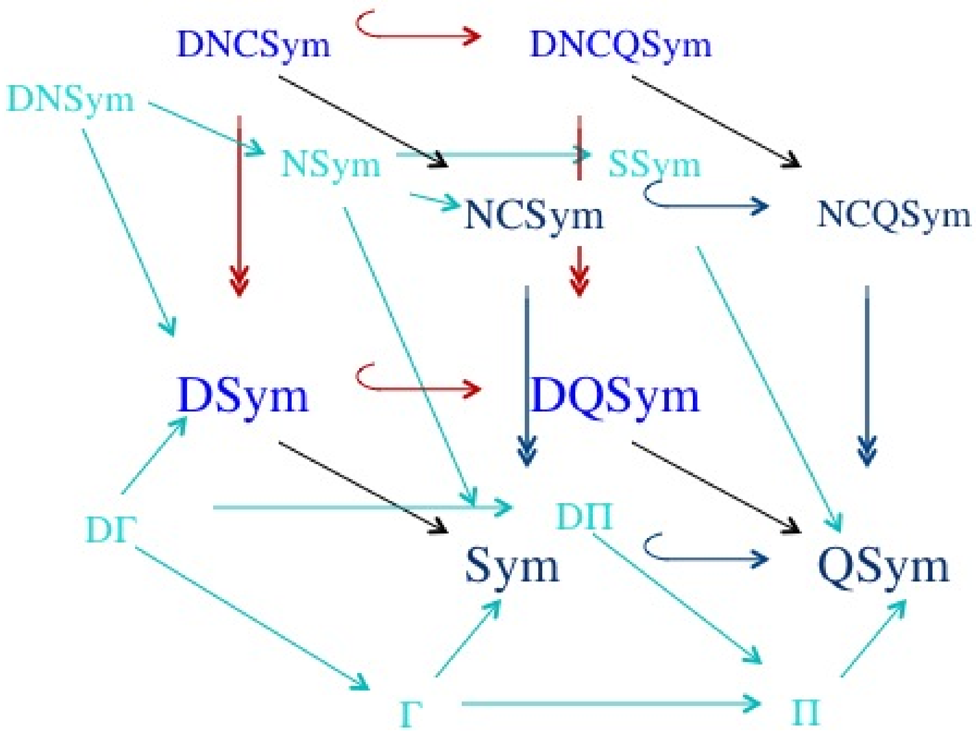
Realization Quotients



3

Jan 2005

Realization Quotients



4

Jan 2005

Diagonally TL-covariants

[Aval Bergeron Bergeron]

$D_n := \mathbb{Q}[x_1, x_2, \ldots, x_n; y_1, \ldots, y_n] / \langle \mathrm{DQSym}^+ \rangle$

Conjectured bigraded Hilbert series:

$$\dim_{qt} D_1 = \begin{bmatrix} 1 \end{bmatrix} \qquad \dim_{qt} D_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\dim_{qt} D_3 = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 2 & 2 \end{bmatrix} \qquad \dim_{qt} D_4 = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 5 & 5 & 0 & 0 \\ 3 & 7 & 5 & 0 \\ 1 & 3 & 5 & 5 \end{bmatrix}$$

+

degree in q (from 0 to n-1)

$$\dim_{qt} D_5 = \begin{bmatrix} 14 & 0 & 0 & 0 & 0 \\ 14 & 14 & 0 & 0 & 0 \\ 9 & 24 & 14 & 0 & 0 \\ 4 & 14 & 24 & 14 & 0 \\ 1 & 4 & 9 & 14 & 14 \end{bmatrix}$$

0 degree in t n-1

5
Jan 2005

Diagonally TL-covariants

[Aval Bergeron Bergeron]

$D_n := Q[x_1, x_2, \ldots, x_n; y_1, \ldots, y_n] / < DQSym^+ >$

Conjectured explicit monomial basis:

for example to build for n=4 and bidegree (1,1)

Start with basis for n=3 →

$$\begin{bmatrix} \{y_3^2, y_2y_3\} & \{\} & \{\} \\ \{y_2, y_3\} & \{y_2x_3, x_3y_3\} & \{\} \\ \{1\} & \{x_2, x_3\} & \{x_3^2, x_3x_2\} \end{bmatrix}$$

$\cdot x_4$ $\cdot x_4y_4$ $\cdot y_4$

Build → $\{x_4y_4, x_2y_4, x_3y_4, x_4y_2, x_4y_3, y_2x_3, x_3y_3\}$

6

Jan 2005

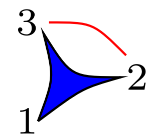
About family of Realization

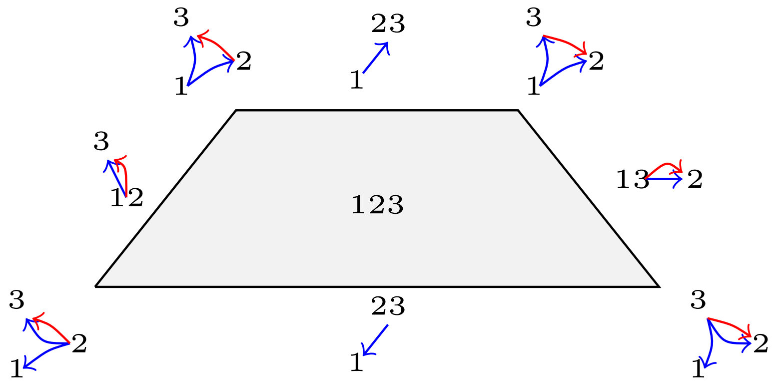
(Q-4) Prove previous question about Hilbert series

(Q-5) Realized Quotient in general

•••

M E R C I





T H A N K S

G R A C I A S