Kirillov-Reshetikhin modules and quantum affine algebras

Bernard Leclerc
Université de Caen

SLC 78
Saint-Nabor, 27/03/2017

Saint-Mélany August 1991

Saint-Mélany August 1991

Saint-Mélany August 1994

SLC 34 Saint-Nabor March 1995

References

[1] E. Verlinde, Nucl. Phys., **B300** (1988), 360.
[2] G. Moore and N. Seiberg, Phys. Lett. (in press).
[3] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys., **B241** (1984), 333.
[4] P. Goddard, A. Kent and D. Olive, Comm. Math. Phys., **103** (1986), 105.
[5] E. Date, M. Jimbo, T. Miwa and M. Okado, Phys. Rev., **B35** (1987), 2105.
[6] J.L. Cardy, Nucl. Phys., **B275** (1986), 200.
[7] H. Saleur and M. Bauer, Nucl. Phys. (in press).
[8] H. Saleur, J. Phys., , **A22** (1989), L41.
[9] J.L. Cardy, Nucl. Phys., **B240[FS12]** (1984), 514.
[10] H.W.J. Blöte, J.L. Cardy and M.P. Nightingale, Phys. Rev. Lett., , **56** (1986), 742.
[11] G.E. Andrews, R.J. Baxter and P.J. Forrester, J. Stat. Phys., **35** (1984), 193.
[12] N. Ishibashi, University of Tokyo preprint UT-530 (1988).
[13] T. Onogi and N. Ishibashi, University of Tokyo preprint UT-Komaba, **88-9** (1988).
[14] H. Sonoda, preprint LBL-25316.
[15] I. Peschel and T.T. Truong, Z. Phys., **B69** (1987), 395.
[16] J.L. Cardy and I. Peschel, Nucl. Phys., **B300[FS22]** (1988), 377.
[17] A. Cappelli, C. Itzykson and J.-B. Zuber, Nucl. Phys., **B280** (1987), 445.
[18] D. Huse, Phys. Rev., **B**, **30** (1984), 3908.
[19] R. J. Baxter, "Exactly Solved Models in Statistical Mechanics", Academic, 1982.
[20] H. Itoyama and H. Thacker, Phys. Rev. Lett., , **14** (1987), 1395.
[21] J.L. Cardy, in "Fields, Strings and Critical Phenomena", Les Houches 1988, E. Brézin and J. Zinn-Justin eds., North-Holland, 1989.
[22] V. Pasquier and H. Saleur, in "Fields, Strings and Critical Phenomena", Les Houches 1988, E. Brézin and J. Zinn-Justin eds., North-Holland, 1989.

Department of Physics
University of California
Santa Barbara CA 93106
U.S.A.

Advanced Studies in Pure Mathematics 19, 1989
Integrable Systems in Quantum Field Theory and Statistical Mechanics
pp. 149–191

Paths, Maya Diagrams and representations of $\widehat{\mathfrak{sl}}(r, \mathbf{C})$

Etsuro Date, Michio Jimbo, Atsuo Kuniba,
Tetsuji Miwa and Masato Okado

Dedicated to Professor Tosihusa Kimura on his 60th birthday

§1. Introduction

Let $\mathfrak{g}$ be the affine Lie algebra $\widehat{\mathfrak{sl}}(r, \mathbf{C})$, let Λ be a dominant integral weight, and let $L(\Lambda)$ be the irreducible $\mathfrak{g}$-module with highest weight Λ. In this article we construct an explicit basis of each weight space $L(\Lambda)_\mu$. As a corollary we prove a new combinatorial formula for the dimensionality of $L(\Lambda)_\mu$, which was conjectured in [1] through the study of corner transfer matrices of solvable lattice models (see Theorem 1.2 below).

The problem of constructing explicit bases goes back to the work of Gelfand and Tsetlin [2] who gave a canonical basis of $L(\Lambda)$ for the classical Lie algebras $\mathfrak{g} = \mathfrak{gl}(r, \mathbf{C})$, $\mathfrak{o}(r, \mathbf{C})$. Analogous results are available in the setting of affine Lie algebras. When Λ is of level 1, $L(\Lambda)$ can be identified with a space of polynomials in infinitely many variables [3,4] or a simple modification thereof [5]. For higher levels, the Z-algebra approach initiated by Lepowsky and Wilson [6] provides a basis in various cases ($\mathfrak{g} = \widehat{\mathfrak{sl}}(2, \mathbf{C})$, arbitrary levels [3],[7], or $\mathfrak{g} = \widehat{\mathfrak{gl}}(r, \mathbf{C}), \widehat{\mathfrak{sp}}(r, \mathbf{C})$, level 2 [8]). Lakshmibai and Seshadri [9] gave a 'standard monomial basis' for $\widehat{\mathfrak{sl}}(2, \mathbf{C})$ using geometric ideas.

A new feature of our approach is the use of an object—*path*, which we now explain. Let $\epsilon_\mu = (0, \cdots, \overset{\mu-\text{th}}{1}, \cdots, 0)$ $(0 \le \mu < r)$ denote the standard base vectors of $\mathbf{Z}^r$. We extend the suffixes to $\mathbf{Z}$ by $\epsilon_{\mu+r} = \epsilon_\mu$. Fix a positive integer l.

Definition 1.1. A path is a sequence $\eta = \left(\eta(k)\right)_{k\ge 0}$ consisting of elements $\eta(k) \in \mathbf{Z}^r$ of the form $\epsilon_{\mu_1(k)} + \cdots + \epsilon_{\mu_l(k)}$ $(\mu_1(k), \cdots, \mu_l(k) \in \mathbf{Z})$.

Received November 15, 1988.

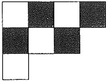
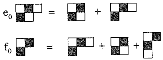


Fig. 2.2 Coloring of nodes.

where α_0, α_1 are the simple roots of $\widehat{\mathfrak{sl}}(2,\mathbf{C})$. Next we define the action of the Chevalley generators e_i, f_i. Put $e_0 Y$ (resp. $f_0 Y$) $= \sum_{Y'} Y'$, where Y' runs over the Young diagrams obtained by removing (resp. adjoining) one white node from Y. For instance,



Likewise define e_1, f_1 replacing 'white' by 'black'. We have then

(2.4) $$(f_i Y, Y') = (Y, e_i Y').$$

With these definitions the irreducible $\widehat{\mathfrak{sl}}(2,\mathbf{C})$-module $L(\Lambda_0)$ is realized as a subspace of $\mathcal{F}[0]$ spanned by vectors of the form $f_{i_1} \cdots f_{i_k} \phi$, ϕ being the empty Young diagram.

There is a natural map $p_{\Lambda_0}: Y \mapsto \eta$ sending the set of Young diagrams onto that of Λ_0-paths. Let Y be a Young diagram, and let g_j denote the length of its $(j+1)$-th column ($j = 0, 1, \cdots, g_j = 0$ for $j \gg 0$). Then $\eta = p_{\Lambda_0}(Y)$ is defined by

$$\eta(j) \in \{0,1\}, \quad \eta(j) \equiv j - g_j \bmod 2 \ (j \geq 0).$$

For instance,

$Y =$  gives $\eta = 1, 0, 1, 1, 0, 1, \cdots$.

Conversely, for each η there exists a unique Young diagram $Y = Y_\eta$

which satisfies the conditions

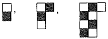
(2.5a) $p_{\Lambda_0}(Y) = \eta$,

(2.5b) Y has the signature $[y_1, y_2, \cdots, y_s]$ with $y_1 > y_2 > \cdots > y_s$.

Thus by (2.5b)

ϕ,  , ... are allowed

but

 , ... are not.

The Young diagram Y_η is called the highest lift of η. It has the property that, for any Y' such that $p_{\Lambda_0}(Y') = \eta$, one has $Y_\eta \subset Y'$.

Our base vectors $\xi_\eta \in L(\Lambda_0)^*$ are defined to be

$$\xi_\eta(v) = (Y_\eta, v), \qquad v \in L(\Lambda_0).$$

Each ξ_η is a weight vector. In the Young diagram picture Y_η, its weight λ_η is simply given by counting the numbers of white and black nodes (2.3). In the path picture η we have

(2.6) $$\lambda_\eta = \mu(0) - \sum_{k\geq 1} k\Big(H\big(\eta(k-1), \eta(k)\big) - H\big(\eta_\Lambda(k-1), \eta_\Lambda(k)\big)\Big)\delta,$$

where $\mu(0)$ is the 'initial point' of the sequence μ (2.2) corresponding to η, $\delta = \alpha_0 + \alpha_1$, and

(2.7) $$\begin{aligned} H(\eta, \eta') &= 0 \quad \text{if } \eta = 0, \eta' = 1, \\ &= 1 \quad \text{otherwise.} \end{aligned}$$

For example, $\eta = \eta^{(3)}$ in (2.1) has the weight $\lambda_\eta = -\Lambda_0 + 2\Lambda_1 - 3\delta$.

One can also construct a basis $\{v_\eta\}$ of $L(\Lambda_0)$ as follows. Consider the process of removing the nodes from Y_η one by one. At each step we require:

(i) removal of the node produces a Young diagram satisfying (2.5b),

(ii) among the nodes satisfying (i) the rightmost one is removed.

Finite-dimensional representations of $\mathfrak{g}$

- $\mathfrak{g}$ simple complex Lie algebra, with Cartan matrix $C = [c_{ij}]_{i,j \in I}$.

Finite-dimensional representations of $\mathfrak{g}$

- $\mathfrak{g}$ simple complex Lie algebra, with Cartan matrix $C = [c_{ij}]_{i,j \in I}$.
- $P = \bigoplus_{i \in I} \mathbb{Z}\varpi_i$ weight lattice of $\mathfrak{g}$.

Finite-dimensional representations of $\mathfrak{g}$

- $\mathfrak{g}$ simple complex Lie algebra, with Cartan matrix $C = [c_{ij}]_{i,j\in I}$.
- $P = \bigoplus_{i\in I} \mathbb{Z}\varpi_i$ weight lattice of $\mathfrak{g}$.
- Simple finite-dimensional $\mathfrak{g}$-modules $L(\lambda)$ are labelled by

$$\lambda \in P_+ = \bigoplus_{i\in I} \mathbb{N}\varpi_i$$

Finite-dimensional representations of $\mathfrak{g}$

- $\mathfrak{g}$ simple complex Lie algebra, with Cartan matrix $C = [c_{ij}]_{i,j \in I}$.
- $P = \bigoplus_{i \in I} \mathbb{Z}\varpi_i$ weight lattice of $\mathfrak{g}$.
- Simple finite-dimensional $\mathfrak{g}$-modules $L(\lambda)$ are labelled by

$$\lambda \in P_+ = \bigoplus_{i \in I} \mathbb{N}\varpi_i$$

- Finite-dimensional $\mathfrak{g}$-modules have a weight-space decomposition

$$M = \bigoplus_{\mu \in P} M_\mu$$

Finite-dimensional representations of $\mathfrak{g}$

- $\mathfrak{g}$ simple complex Lie algebra, with Cartan matrix $C = [c_{ij}]_{i,j\in I}$.
- $P = \bigoplus_{i\in I} \mathbb{Z}\varpi_i$ weight lattice of $\mathfrak{g}$.
- Simple finite-dimensional $\mathfrak{g}$-modules $L(\lambda)$ are labelled by

$$\lambda \in P_+ = \bigoplus_{i\in I} \mathbb{N}\varpi_i$$

- Finite-dimensional $\mathfrak{g}$-modules have a weight-space decomposition

$$M = \bigoplus_{\mu\in P} M_\mu$$

$\rightsquigarrow$ character of M: $\chi(M) := \sum_{\mu\in P} \dim(M_\mu)\, e^\mu$

Finite-dimensional representations of $\mathfrak{g}$

- $\mathfrak{g}$ simple complex Lie algebra, with Cartan matrix $C = [c_{ij}]_{i,j\in I}$.
- $P = \bigoplus_{i\in I} \mathbb{Z}\varpi_i$ weight lattice of $\mathfrak{g}$.
- Simple finite-dimensional $\mathfrak{g}$-modules $L(\lambda)$ are labelled by

$$\lambda \in P_+ = \bigoplus_{i\in I} \mathbb{N}\varpi_i$$

- Finite-dimensional $\mathfrak{g}$-modules have a weight-space decomposition

$$M = \bigoplus_{\mu\in P} M_\mu$$

$\rightsquigarrow$ character of M: $\chi(M) := \sum_{\mu\in P} \dim(M_\mu)\, e^\mu$

- For $\lambda \in P_+$, $\chi(L(\lambda))$ is given by Weyl's character formula.

Characters

Characters

- $\chi(M)$ is a Laurent polynomial in the $y_i := e^{\varpi_i}$.

Characters

- $\chi(M)$ is a Laurent polynomial in the $y_i := e^{\varpi_i}$.

Example: $\mathfrak{g} = \mathfrak{sl}_3$ (type A_2).

Characters

- $\chi(M)$ is a Laurent polynomial in the $y_i := e^{\varpi_i}$.

Example: $\mathfrak{g} = \mathfrak{sl}_3$ (type A_2).

$$\begin{aligned} \chi(L(\varpi_1)) &= y_1 + y_1^{-1} y_2 + y_2^{-1} \\ \chi(L(\varpi_2)) &= y_2 + y_2^{-1} y_1 + y_1^{-1} \end{aligned}$$

Characters

- $\chi(M)$ is a Laurent polynomial in the $y_i := e^{\varpi_i}$.

Example: $\mathfrak{g} = \mathfrak{sl}_3$ (type A_2).

$$\begin{aligned} \chi(L(\varpi_1)) &= y_1 + y_1^{-1} y_2 + y_2^{-1} \\ \chi(L(\varpi_2)) &= y_2 + y_2^{-1} y_1 + y_1^{-1} \end{aligned}$$

Example: $\mathfrak{g} = \mathfrak{so}_5$ (type B_2).

Characters

- $\chi(M)$ is a Laurent polynomial in the $y_i := e^{\varpi_i}$.

Example: $\mathfrak{g} = \mathfrak{sl}_3$ (type A_2).

$$\begin{aligned}\chi(L(\varpi_1)) &= y_1 + y_1^{-1} y_2 + y_2^{-1} \\ \chi(L(\varpi_2)) &= y_2 + y_2^{-1} y_1 + y_1^{-1}\end{aligned}$$

Example: $\mathfrak{g} = \mathfrak{so}_5$ (type B_2).

$$\begin{aligned}\chi(L(\varpi_1)) &= y_1 + y_1^{-1} y_2^2 + 1 + y_1 y_2^{-2} + y_1^{-1} \\ \chi(L(\varpi_2)) &= y_2 + y_1 y_2^{-1} + y_1^{-1} y_2 + y_2^{-1}\end{aligned}$$

Finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$

Finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$

- $\mathscr{C}$ category of finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules.

($q \in \mathbb{C}^*$, not a root of 1)

Finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$

- $\mathscr{C}$ category of finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules.
  ($q \in \mathbb{C}^*$, not a root of 1)
- Lattice of ℓ-weights: $\widehat{P} := \bigoplus_{z \in \mathbb{C}^*} \bigoplus_{i \in I} \mathbb{Z}(\varpi_i, z)$.

Finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$

- $\mathscr{C}$ category of finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules.

($q \in \mathbb{C}^*$, not a root of 1)

- Lattice of ℓ-weights: $\widehat{P} := \bigoplus_{z \in \mathbb{C}^*} \bigoplus_{i \in I} \mathbb{Z}(\varpi_i, z)$.

Theorem (Chari-Pressley)

- Simples of $\mathscr{C}$ are parametrized by $\widehat{P}_+ = \bigoplus_{i,z} \mathbb{N}(\varpi_i, z)$.

Finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$

- $\mathscr{C}$ category of finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules.

($q \in \mathbb{C}^*$, not a root of 1)

- Lattice of ℓ-weights: $\widehat{P} := \bigoplus_{z \in \mathbb{C}^*} \bigoplus_{i \in I} \mathbb{Z}(\varpi_i, z)$.

Theorem (Chari-Pressley)

- Simples of $\mathscr{C}$ are parametrized by $\widehat{P}_+ = \bigoplus_{i,z} \mathbb{N}(\varpi_i, z)$.
- $K_0(\mathscr{C}) =$ polynomial ring in the $[L(\varpi_i, z)]$ ($i \in I$, $z \in \mathbb{C}^*$).

Finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$

- $\mathscr{C}$ category of finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules.
($q \in \mathbb{C}^*$, not a root of 1)
- Lattice of ℓ-weights: $\widehat{P} := \bigoplus_{z \in \mathbb{C}^*} \bigoplus_{i \in I} \mathbb{Z}(\varpi_i, z)$.

Theorem (Chari-Pressley)

- Simples of $\mathscr{C}$ are parametrized by $\widehat{P}_+ = \bigoplus_{i,z} \mathbb{N}(\varpi_i, z)$.
- $K_0(\mathscr{C}) =$ polynomial ring in the $[L(\varpi_i, z)]$ ($i \in I$, $z \in \mathbb{C}^*$).

- $M \in \mathscr{C}$ has an ℓ-weight-space decomposition $M = \bigoplus_{\widehat{\lambda} \in \widehat{P}} M_{\widehat{\lambda}}$.

Finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$

- $\mathscr{C}$ category of finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules.

($q \in \mathbb{C}^*$, not a root of 1)

- Lattice of ℓ-weights: $\widehat{P} := \bigoplus_{z \in \mathbb{C}^*} \bigoplus_{i \in I} \mathbb{Z}(\varpi_i, z)$.

Theorem (Chari-Pressley)

- Simples of $\mathscr{C}$ are parametrized by $\widehat{P}_+ = \bigoplus_{i,z} \mathbb{N}(\varpi_i, z)$.
- $K_0(\mathscr{C}) =$ polynomial ring in the $[L(\varpi_i, z)]$ ($i \in I$, $z \in \mathbb{C}^*$).

- $M \in \mathscr{C}$ has an ℓ-weight-space decomposition $M = \bigoplus_{\widehat{\lambda} \in \widehat{P}} M_{\widehat{\lambda}}$.

Definition (Frenkel-Reshetikhin)

$$\chi_q(M) := \sum_{\widehat{\lambda} \in \widehat{P}} \dim M_{\widehat{\lambda}}\, e^{\widehat{\lambda}} = q\text{-character of } M.$$

Finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$

- $\mathscr{C}$ category of finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules.

($q \in \mathbb{C}^*$, not a root of 1)

- Lattice of ℓ-weights: $\widehat{P} := \bigoplus_{z\in\mathbb{C}^*} \bigoplus_{i\in I} \mathbb{Z}(\varpi_i, z)$.

Theorem (Chari-Pressley)

- Simples of $\mathscr{C}$ are parametrized by $\widehat{P}_+ = \bigoplus_{i,z} \mathbb{N}(\varpi_i, z)$.
- $K_0(\mathscr{C})$ = polynomial ring in the $[L(\varpi_i, z)]$ ($i \in I$, $z \in \mathbb{C}^*$).

- $M \in \mathscr{C}$ has an ℓ-weight-space decomposition $M = \bigoplus_{\widehat{\lambda}\in\widehat{P}} M_{\widehat{\lambda}}$.

Definition (Frenkel-Reshetikhin)

$$\chi_q(M) := \sum_{\widehat{\lambda}\in\widehat{P}} \dim M_{\widehat{\lambda}}\, e^{\widehat{\lambda}} = q\text{-character of } M.$$

- $[M] = [N]$ in $K_0(\mathscr{C}) \iff \chi_q(M) = \chi_q(N)$.

q-characters

q-characters

- $\chi_q(M)$ is a Laurent polynomial in the $Y_{i,z} := e^{(\varpi_i, z)}$.

q-characters

- $\chi_q(M)$ is a Laurent polynomial in the $Y_{i,z} := e^{(\varpi_i, z)}$.

Example: $\mathfrak{g} = \mathfrak{sl}_3$ (type A_2).

q-characters

- $\chi_q(M)$ is a Laurent polynomial in the $Y_{i,z} := e^{(\varpi_i, z)}$.

Example: $\mathfrak{g} = \mathfrak{sl}_3$ (type A_2).

$$\chi_q(L(\varpi_1, z)) = Y_{1,z} + Y_{1,zq^2}^{-1} Y_{2,zq} + Y_{2,zq^3}^{-1}$$

q-characters

- $\chi_q(M)$ is a Laurent polynomial in the $Y_{i,z} := e^{(\varpi_i, z)}$.

Example: $\mathfrak{g} = \mathfrak{sl}_3$ (type A_2).

$$\chi_q(L(\varpi_1, z)) = Y_{1,z} + Y_{1,zq^2}^{-1} Y_{2,zq} + Y_{2,zq^3}^{-1}$$

Example: $\mathfrak{g} = \mathfrak{so}_5$ (type B_2).

q-characters

- $\chi_q(M)$ is a Laurent polynomial in the $Y_{i,z} := e^{(\varpi_i, z)}$.

Example: $\mathfrak{g} = \mathfrak{sl}_3$ (type A_2).

$$\chi_q(L(\varpi_1, z)) = Y_{1,z} + Y^{-1}_{1,zq^2} Y_{2,zq} + Y^{-1}_{2,zq^3}$$

Example: $\mathfrak{g} = \mathfrak{so}_5$ (type B_2).

$$\begin{aligned}\chi_q(L(\varpi_1, z)) &= Y_{1,z} + Y^{-1}_{1,zq^4} Y_{2,zq} Y_{2,zq^3} + Y_{2,zq} Y^{-1}_{2,zq^5} \\ &\quad + Y_{1,zq^2} Y^{-1}_{2,zq^3} Y^{-1}_{2,zq^5} + Y^{-1}_{1,zq^6} \\ \chi_q(L(\varpi_2, z)) &= Y_{2,z} + Y_{1,zq} Y^{-1}_{2,zq^2} + Y^{-1}_{1,zq^5} Y_{2,zq^4} + Y^{-1}_{2,zq^6}\end{aligned}$$

q-characters

- $\chi_q(M)$ is a Laurent polynomial in the $Y_{i,z} := e^{(\varpi_i, z)}$.

Example: $\mathfrak{g} = \mathfrak{sl}_3$ (type A_2).

$$\chi_q(L(\varpi_1, z)) = Y_{1,z} + Y_{1,zq^2}^{-1} Y_{2,zq} + Y_{2,zq^3}^{-1}$$

Example: $\mathfrak{g} = \mathfrak{so}_5$ (type B_2).

$$\begin{aligned} \chi_q(L(\varpi_1, z)) &= Y_{1,z} + Y_{1,zq^4}^{-1} Y_{2,zq} Y_{2,zq^3} + Y_{2,zq} Y_{2,zq^5}^{-1} \\ &\quad + Y_{1,zq^2} Y_{2,zq^3}^{-1} Y_{2,zq^5}^{-1} + Y_{1,zq^6}^{-1} \\ \chi_q(L(\varpi_2, z)) &= Y_{2,z} + Y_{1,zq} Y_{2,zq^2}^{-1} + Y_{1,zq^5}^{-1} Y_{2,zq^4} + Y_{2,zq^6}^{-1} \end{aligned}$$

- There is no Weyl type character formula for $\chi_q\left(L(\widehat{\lambda})\right)$.

q-characters

- $\chi_q(M)$ is a Laurent polynomial in the $Y_{i,z} := e^{(\varpi_i, z)}$.

Example: $\mathfrak{g} = \mathfrak{sl}_3$ (type A_2).

$$\chi_q(L(\varpi_1, z)) = Y_{1,z} + Y^{-1}_{1,zq^2} Y_{2,zq} + Y^{-1}_{2,zq^3}$$

Example: $\mathfrak{g} = \mathfrak{so}_5$ (type B_2).

$$\begin{aligned}
\chi_q(L(\varpi_1, z)) &= Y_{1,z} + Y^{-1}_{1,zq^4} Y_{2,zq} Y_{2,zq^3} + Y_{2,zq} Y^{-1}_{2,zq^5} \\
&\quad + Y_{1,zq^2} Y^{-1}_{2,zq^3} Y^{-1}_{2,zq^5} + Y^{-1}_{1,zq^6} \\
\chi_q(L(\varpi_2, z)) &= Y_{2,z} + Y_{1,zq} Y^{-1}_{2,zq^2} + Y^{-1}_{1,zq^5} Y_{2,zq^4} + Y^{-1}_{2,zq^6}
\end{aligned}$$

- There is no Weyl type character formula for $\chi_q\left(L(\widehat{\lambda})\right)$. But
  - Frenkel-Mukhin's algorithm for minuscule representations.

q-characters

- $\chi_q(M)$ is a Laurent polynomial in the $Y_{i,z} := e^{(\varpi_i, z)}$.

Example: $\mathfrak{g} = \mathfrak{sl}_3$ (type A_2).

$$\chi_q(L(\varpi_1, z)) = Y_{1,z} + Y_{1,zq^2}^{-1} Y_{2,zq} + Y_{2,zq^3}^{-1}$$

Example: $\mathfrak{g} = \mathfrak{so}_5$ (type B_2).

$$\begin{aligned} \chi_q(L(\varpi_1, z)) &= Y_{1,z} + Y_{1,zq^4}^{-1} Y_{2,zq} Y_{2,zq^3} + Y_{2,zq} Y_{2,zq^5}^{-1} \\ &\quad + Y_{1,zq^2} Y_{2,zq^3}^{-1} Y_{2,zq^5}^{-1} + Y_{1,zq^6}^{-1} \\ \chi_q(L(\varpi_2, z)) &= Y_{2,z} + Y_{1,zq} Y_{2,zq^2}^{-1} + Y_{1,zq^5}^{-1} Y_{2,zq^4} + Y_{2,zq^6}^{-1} \end{aligned}$$

- There is no Weyl type character formula for $\chi_q\left(L(\widehat{\lambda})\right)$. But
  - Frenkel-Mukhin's algorithm for minuscule representations.
  - Nakajima's geometric description for $\mathfrak{g}$ of type A, D, E.

Kirillov-Reshetikhin modules

Kirillov-Reshetikhin modules

Notation: For $i \in I$, $k \in \mathbb{N}$, and $z \in \mathbb{C}^*$ put

$$W_{k,z}^{(i)} := L\Big((\varpi_i, z) + (\varpi_i, zq^2) + \cdots + (\varpi_i, zq^{2k-2})\Big).$$

Kirillov-Reshetikhin modules

Notation: For $i \in I$, $k \in \mathbb{N}$, and $z \in \mathbb{C}^*$ put

$$W_{k,z}^{(i)} := L\left((\varpi_i, z) + (\varpi_i, zq^2) + \cdots + (\varpi_i, zq^{2k-2})\right).$$

Theorem (Kuniba-Nakanishi-Suzuki, Nakajima, Hernandez)

The $[W_{k,z}^{(i)}]$ satisfy an infinite system of equations, called T-system.

Kirillov-Reshetikhin modules

Notation: For $i \in I$, $k \in \mathbb{N}$, and $z \in \mathbb{C}^*$ put

$$W_{k,z}^{(i)} := L\Big((\varpi_i, z) + (\varpi_i, zq^2) + \cdots + (\varpi_i, zq^{2k-2})\Big).$$

Theorem (Kuniba-Nakanishi-Suzuki, Nakajima, Hernandez)

The $[W_{k,z}^{(i)}]$ satisfy an infinite system of equations, called T-system.
In type A, D, E:

$$\left[W_{k,z}^{(i)}\right]\left[W_{k,zq^2}^{(i)}\right] = \left[W_{k+1,z}^{(i)}\right]\left[W_{k-1,zq^2}^{(i)}\right] + \prod_{j \neq i}\left[W_{k,zq}^{(j)}\right]^{-c_{ij}}$$

Infinite quivers attached to Cartan matrices

Infinite quivers attached to Cartan matrices

- $C = (c_{ij} \mid i, j \in I)$, Cartan matrix $(A_n, B_n, \ldots, F_4, G_2)$

Infinite quivers attached to Cartan matrices

- $C = (c_{ij} \mid i, j \in I)$, Cartan matrix $(A_n, B_n, \ldots, F_4, G_2)$
- $D = \mathrm{diag}(d_i)$, $d_i \in \mathbb{Z}_{>0}$, $\min(d_i) = 1$, such that DC is symmetric

Infinite quivers attached to Cartan matrices

- $C = (c_{ij} \mid i, j \in I)$, Cartan matrix $(A_n, B_n, \ldots, F_4, G_2)$
- $D = \mathrm{diag}(d_i)$, $d_i \in \mathbb{Z}_{>0}$, $\min(d_i) = 1$, such that DC is symmetric
- $\widetilde{Q}$, quiver with vertex set $I \times \mathbb{Z}$, and arrows:

$$(i, r) \to (j, s) \quad \Longleftrightarrow \quad c_{ij} \neq 0 \text{ and } s = r + d_i c_{ij}$$

Infinite quivers attached to Cartan matrices

- $C = (c_{ij} \mid i, j \in I)$, Cartan matrix $(A_n, B_n, \dots, F_4, G_2)$
- $D = \mathrm{diag}(d_i)$, $d_i \in \mathbb{Z}_{>0}$, $\min(d_i) = 1$, such that DC is symmetric
- $\widetilde{Q}$, quiver with vertex set $I \times \mathbb{Z}$, and arrows:

$$(i, r) \to (j, s) \quad \Longleftrightarrow \quad c_{ij} \neq 0 \text{ and } s = r + d_i c_{ij}$$

- Q, one of the two connected components of $\widetilde{Q}$

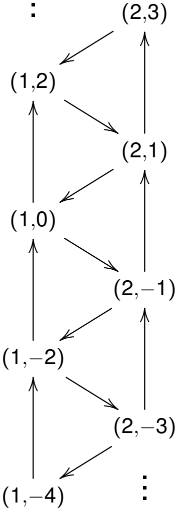
Type A_2

Type A_2

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Type A_2

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$



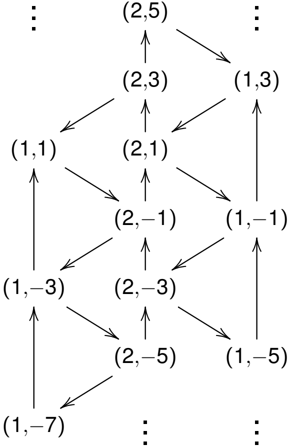
Type B_2

Type B_2

$$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

Type B_2

$$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$



Semi-infinite quivers

Semi-infinite quivers

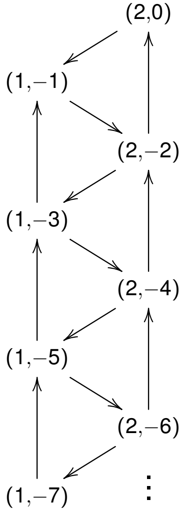
- infinite quiver Γ : same graph as Q, but change of vertex labelling: (i, r) is changed into $(i, r + d_i)$. New vertex set W.

Semi-infinite quivers

- infinite quiver Γ : same graph as Q, but change of vertex labelling: (i, r) is changed into $(i, r + d_i)$. New vertex set W.
- semi-infinite quiver Γ^-, fullsubquiver of Γ with vertex set $W^- = \{(i, s) \in W \mid s \leq 0\}$.

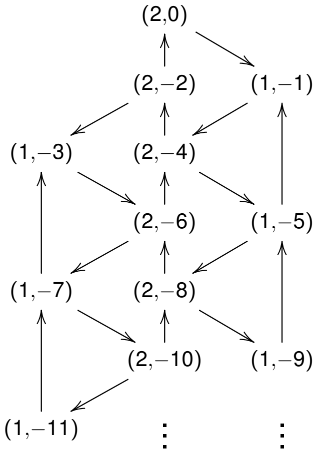
Type A_2

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$



Type B_2

$$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$



q-characters of KR-modules

q-characters of KR-modules

- $z := \{z_{i,s} \mid (i,s) \in W^-\}$, set of indeterminates

q-characters of KR-modules

- $z := \{z_{i,s} \mid (i,s) \in W^-\}$, set of indeterminates

Definition

$\mathscr{A}$, cluster algebra with initial seed (z, Γ^-)

q-characters of KR-modules

- $z := \{z_{i,s} \mid (i,s) \in W^-\}$, set of indeterminates

Definition

$\mathscr{A}$, cluster algebra with initial seed (z, Γ^-)

- monomial change of variables : $z_{i,r} = \prod_{k>0, r+kd_i \leq 0} Y_{i,q^{r+kd_i}}$

q-characters of KR-modules

- $z := \{z_{i,s} \mid (i,s) \in W^-\}$, set of indeterminates

Definition

$\mathscr{A}$, cluster algebra with initial seed (z, Γ^-)

- monomial change of variables : $z_{i,r} = \prod_{k>0, r+kd_i \leq 0} Y_{i,q^{r+kd_i}}$
- $h^\vee$ dual Coxeter number of $\mathfrak{g}$.

q-characters of KR-modules

- $z := \{z_{i,s} \mid (i,s) \in W^-\}$, set of indeterminates

Definition

$\mathscr{A}$, cluster algebra with initial seed (z, Γ^-)

- monomial change of variables : $z_{i,r} = \prod_{k>0, r+kd_i \leq 0} Y_{i,q^{r+kd_i}}$
- $h^\vee$ dual Coxeter number of $\mathfrak{g}$.

Theorem (Hernandez-L)

There exists an explicit sequence of mutations $\mathscr{S}$ such that:

q-characters of KR-modules

- $z := \{z_{i,s} \mid (i,s) \in W^-\}$, set of indeterminates

Definition

$\mathscr{A}$, cluster algebra with initial seed (z, Γ^-)

- monomial change of variables : $z_{i,r} = \prod_{k>0, r+kd_i \leq 0} Y_{i,q^{r+kd_i}}$
- $h^\vee$ dual Coxeter number of $\mathfrak{g}$.

Theorem (Hernandez-L)

There exists an explicit sequence of mutations $\mathscr{S}$ such that:

- $\mu_{\mathscr{S}}(\Sigma)$ has the same quiver as Σ

q-characters of KR-modules

- $z := \{z_{i,s} \mid (i,s) \in W^-\}$, set of indeterminates

Definition

$\mathscr{A}$, cluster algebra with initial seed (z, Γ^-)

- monomial change of variables : $z_{i,r} = \prod_{k>0, r+kd_i \leq 0} Y_{i,q^{r+kd_i}}$
- $h^\vee$ dual Coxeter number of $\mathfrak{g}$.

Theorem (Hernandez-L)

There exists an explicit sequence of mutations $\mathscr{S}$ such that:

- $\mu_{\mathscr{S}}(\Sigma)$ has the same quiver as Σ
- For $m \geq h^\vee/2$, the cluster variables of $\mu_{\mathscr{S}}^m(\Sigma)$ are the q-characters of all KR-modules (up to spectral shift)

q-characters of KR-modules

- $z := \{z_{i,s} \mid (i,s) \in W^-\}$, set of indeterminates

Definition

$\mathscr{A}$, cluster algebra with initial seed (z, Γ^-)

- monomial change of variables : $z_{i,r} = \prod_{k>0, r+kd_i \leq 0} Y_{i,q^{r+kd_i}}$
- $h^\vee$ dual Coxeter number of $\mathfrak{g}$.

Theorem (Hernandez-L)

There exists an explicit sequence of mutations $\mathscr{S}$ such that:

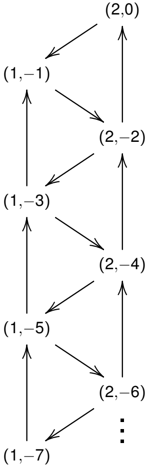
- $\mu_{\mathscr{S}}(\Sigma)$ has the same quiver as Σ
- For $m \geq h^\vee/2$, the cluster variables of $\mu_{\mathscr{S}}^m(\Sigma)$ are the q-characters of all KR-modules (up to spectral shift)

$\rightsquigarrow$ algorithm to calculate q-characters of KR-modules by "successive approximations".

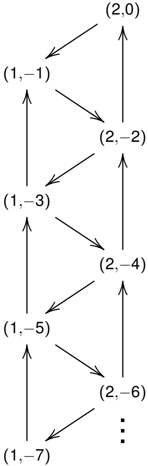
The rules are tricky, but they are a much more efficient way of getting the answer than by counting beans.

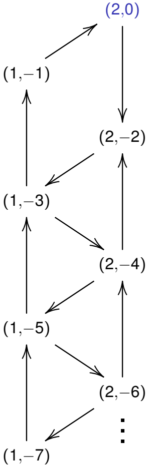
Richard Feynman, *QED the strange theory of light and matter,* 1985.

The sequence $\mathscr{S}$: type A_2

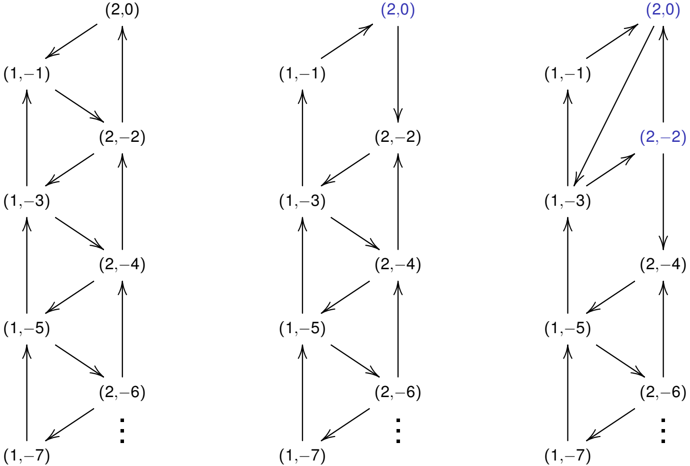


The sequence $\mathscr{S}$: type A_2

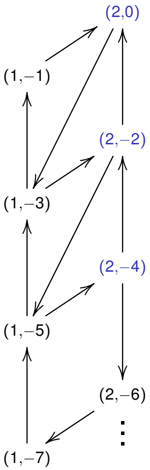




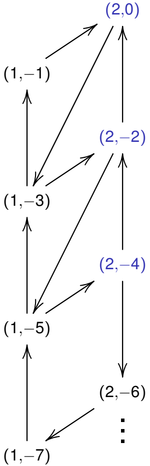
The sequence $\mathscr{S}$: type A_2

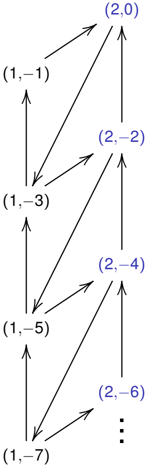


The sequence $\mathscr{S}$: type A_2

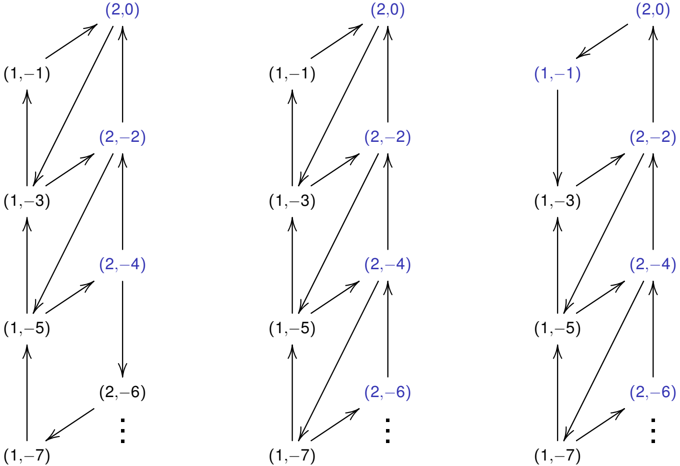


The sequence $\mathscr{S}$: type A_2

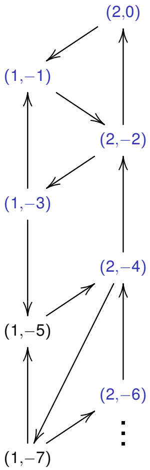




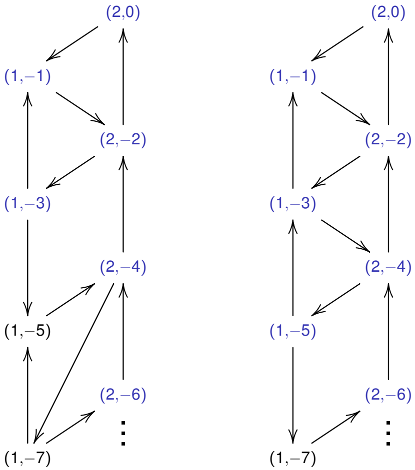
The sequence $\mathscr{S}$: type A_2



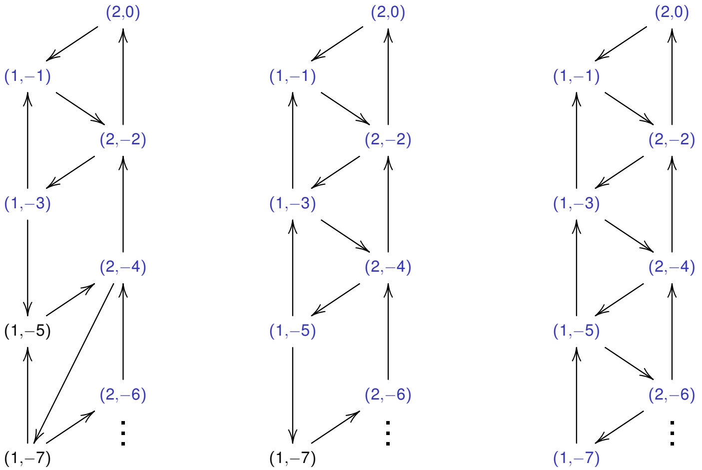
The sequence $\mathscr{S}$: type A_2



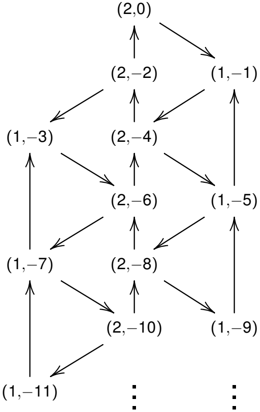
The sequence $\mathscr{S}$: type A_2



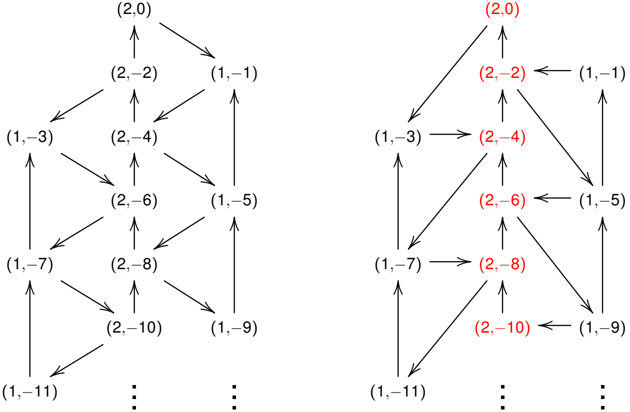
The sequence $\mathscr{S}$: type A_2



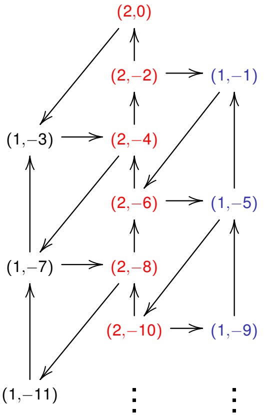
The sequence $\mathscr{S}$: type B_2



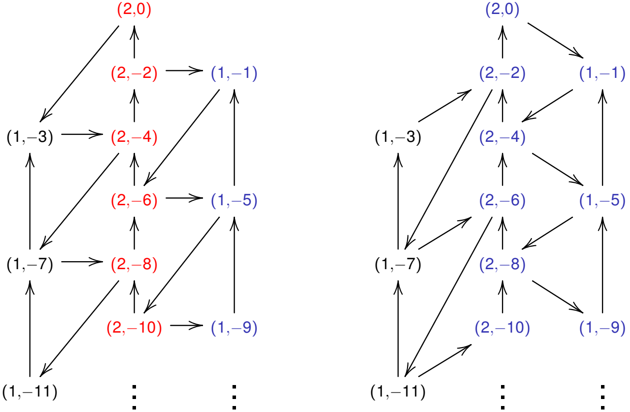
The sequence $\mathscr{S}$: type B_2



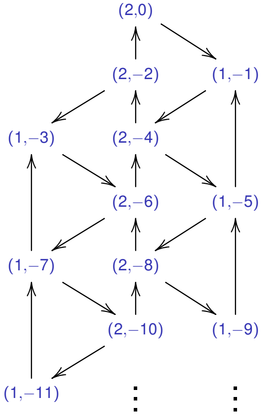
The sequence $\mathscr{S}$: type B_2



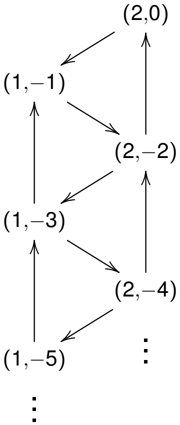
The sequence $\mathscr{S}$: type B_2



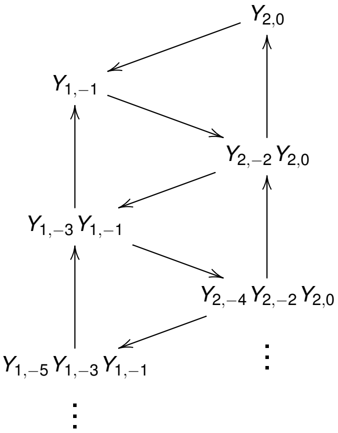
The sequence $\mathscr{S}$: type B_2



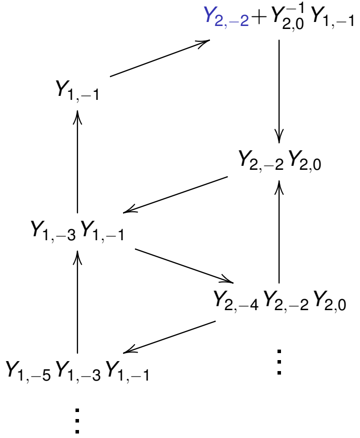
The algorithm : type A_2



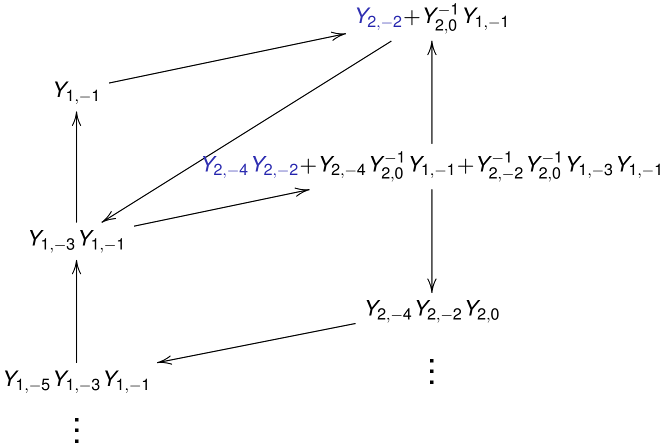
The algorithm : type A_2



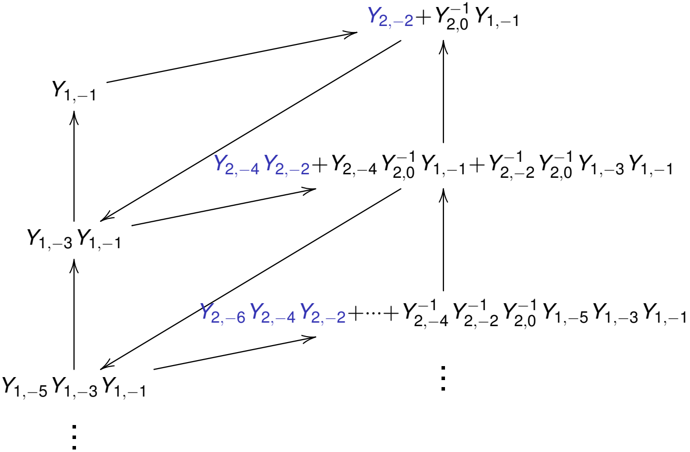
The algorithm : type A_2



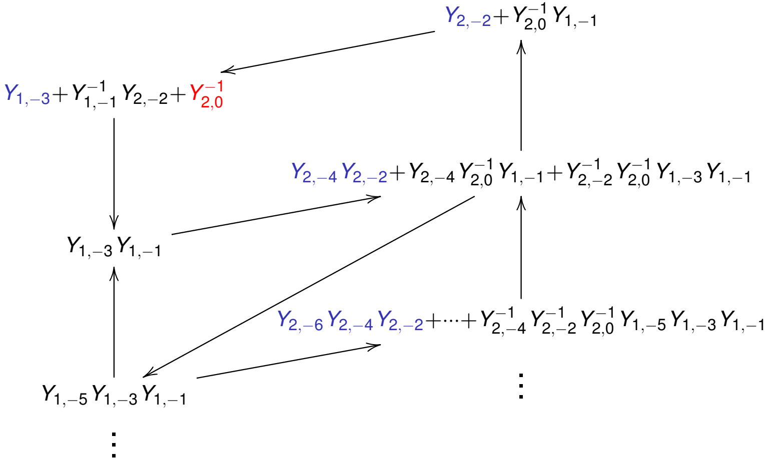
The algorithm : type A_2



The algorithm : type A_2



The algorithm : type A_2



The algorithm : type A_2

$$Y_{2,-2}+Y_{2,0}^{-1}Y_{1,-1}$$

$$Y_{1,-3}+Y_{1,-1}^{-1}Y_{2,-2}+Y_{2,0}^{-1}$$

$$Y_{2,-4}Y_{2,-2}+Y_{2,-4}Y_{2,0}^{-1}Y_{1,-1}+Y_{2,-2}^{-1}Y_{2,0}^{-1}Y_{1,-3}Y_{1,-1}$$

$$Y_{1,-5}Y_{1,-3}+\cdots+Y_{2,-2}^{-1}Y_{2,0}^{-1}$$

$$Y_{2,-6}Y_{2,-4}Y_{2,-2}+\cdots+Y_{2,-4}^{-1}Y_{2,-2}^{-1}Y_{2,0}^{-1}Y_{1,-5}Y_{1,-3}Y_{1,-1}$$

$$Y_{1,-5}Y_{1,-3}Y_{1,-1}$$

$$\vdots \qquad \vdots$$

The algorithm : type A_2

$$Y_{2,-2} + Y_{2,0}^{-1} Y_{1,-1}$$

$$Y_{1,-3} + Y_{1,-1}^{-1} Y_{2,-2} + Y_{2,0}^{-1}$$

$$Y_{2,-4} Y_{2,-2} + Y_{2,-4} Y_{2,0}^{-1} Y_{1,-1} + Y_{2,-2}^{-1} Y_{2,0}^{-1} Y_{1,-3} Y_{1,-1}$$

$$Y_{1,-5} Y_{1,-3} + \cdots + Y_{2,-2}^{-1} Y_{2,0}^{-1}$$

$$Y_{2,-6} Y_{2,-4} Y_{2,-2} + \cdots + Y_{2,-4}^{-1} Y_{2,-2}^{-1} Y_{2,0}^{-1} Y_{1,-5} Y_{1,-3} Y_{1,-1}$$

$$Y_{1,-7} Y_{1,-5} Y_{1,-3} + \cdots + Y_{2,-4}^{-1} Y_{2,-2}^{-1} Y_{2,0}^{-1}$$

$$\vdots \qquad \vdots$$

The algorithm : type A_2

$$Y_{2,-4} + Y_{2,-2}^{-1} Y_{1,-3} + Y_{1,-1}^{-1}$$

$$Y_{1,-5} + Y_{1,-3}^{-1} Y_{2,-4} + Y_{2,-2}^{-1}$$

$$Y_{2,-6} Y_{2,-4} + \cdots + Y_{1,-3}^{-1} Y_{1,-1}^{-1}$$

$$Y_{1,-7} Y_{1,-5} + \cdots + Y_{2,-4}^{-1} Y_{2,-2}^{-1}$$

$$Y_{2,-8} Y_{2,-6} Y_{2,-4} + \cdots + Y_{1,-5}^{-1} Y_{1,-3}^{-1} Y_{1,-1}^{-1}$$

$$\vdots$$

$$Y_{1,-9} Y_{1,-7} Y_{1,-5} + \cdots + Y_{2,-6}^{-1} Y_{2,-4}^{-1} Y_{2,-2}^{-1}$$

$$\vdots$$

The CC-DWZ philosophy

The CC-DWZ philosophy

- $\mathscr{A}$ cluster algebra with initial seed $((x_1, \ldots, x_n), Q)$.

The CC-DWZ philosophy

- $\mathscr{A}$ cluster algebra with initial seed $((x_1, \ldots, x_n), Q)$.
- $\rightsquigarrow$ A, an associative algebra of the form $\mathbb{C}Q/I$ where I is an ideal given by a superpotential of Q.

The CC-DWZ philosophy

- $\mathscr{A}$ cluster algebra with initial seed $((x_1, \ldots, x_n), Q)$.
- $\rightsquigarrow A$, an associative algebra of the form $\mathbb{C}Q/I$ where I is an ideal given by a superpotential of Q.
- z, cluster variable of $\mathscr{A}$ $\quad\rightsquigarrow\quad$ M, indecomposable A-module

The CC-DWZ philosophy

- $\mathscr{A}$ cluster algebra with initial seed $((x_1, \ldots, x_n), Q)$.
- $\rightsquigarrow A$, an associative algebra of the form $\mathbb{C}Q/I$ where I is an ideal given by a superpotential of Q.
- z, cluster variable of $\mathscr{A}$ $\rightsquigarrow$ M, indecomposable A-module
- expansion of z in terms of x_i's is encoded in the topology of varieties of submodules of M:

The CC-DWZ philosophy

- $\mathscr{A}$ cluster algebra with initial seed $((x_1, \ldots, x_n), Q)$.
- $\rightsquigarrow A$, an associative algebra of the form $\mathbb{C}Q/I$ where I is an ideal given by a superpotential of Q.
- z, cluster variable of $\mathscr{A}$ $\quad\rightsquigarrow\quad$ M, indecomposable A-module
- expansion of z in terms of x_i's is encoded in the topology of varieties of submodules of M:
    - monomials m $\quad\longleftrightarrow\quad$ dimension vectors d of submodules

The CC-DWZ philosophy

- $\mathscr{A}$ cluster algebra with initial seed $((x_1, \dots, x_n), Q)$.
- $\rightsquigarrow A$, an associative algebra of the form $\mathbb{C}Q/I$ where I is an ideal given by a superpotential of Q.
- z, cluster variable of $\mathscr{A}$ $\quad\rightsquigarrow\quad$ M, indecomposable A-module
- expansion of z in terms of x_i's is encoded in the topology of varieties of submodules of M:
  - monomials $m \quad\longleftrightarrow\quad$ dimension vectors d of submodules
  - coefficient of m = Euler characteristic of the variety of submodules of dimension d

The CC-DWZ philosophy

- $\mathscr{A}$ cluster algebra with initial seed $((x_1, \ldots, x_n), Q)$.
- $\rightsquigarrow A$, an associative algebra of the form $\mathbb{C}Q/I$ where I is an ideal given by a superpotential of Q.
- z, cluster variable of $\mathscr{A}$ $\rightsquigarrow$ M, indecomposable A-module
- expansion of z in terms of x_i's is encoded in the topology of varieties of submodules of M:
  - monomials m $\longleftrightarrow$ dimension vectors d of submodules
  - coefficient of m = Euler characteristic of the variety of submodules of dimension d

Theorem (Hernandez-L)

Geometric character formulas for q-characters of KR-modules

Type A_2

$$\chi_q\left(W^{(1)}_{1,-3}\right) = Y_{1,-3} + Y^{-1}_{1,-1} Y_{2,-2} + Y^{-1}_{2,0}$$

Type A_2

$$
\begin{aligned}
\chi_q\left(W^{(1)}_{1,-3}\right) &= Y_{1,-3} + Y^{-1}_{1,-1} Y_{2,-2} + Y^{-1}_{2,0} \\
&= Y_{1,-3}\left(1 + {A_{1,-2}}^{-1} + {A_{1,-2}}^{-1} {A_{2,-1}}^{-1}\right)
\end{aligned}
$$

Type A_2

$$\chi_q\left(W^{(1)}_{1,-3}\right) = Y_{1,-3} + Y^{-1}_{1,-1} Y_{2,-2} + Y^{-1}_{2,0}$$
$$= Y_{1,-3}\left(1 + {A_{1,-2}}^{-1} + {A_{1,-2}}^{-1} {A_{2,-1}}^{-1}\right)$$

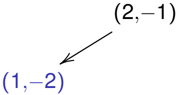
$$A_{1,r} = Y_{1,r-1} Y_{1,r+1} Y^{-1}_{2,r}, \quad A_{2,r} = Y_{2,r-1} Y_{2,r+1} Y^{-1}_{1,r}.$$

Type A_2

$$\chi_q\left(W^{(1)}_{1,-3}\right) = Y_{1,-3} + Y_{1,-1}^{-1} Y_{2,-2} + Y_{2,0}^{-1}$$
$$= Y_{1,-3}\left(1 + {A_{1,-2}}^{-1} + {A_{1,-2}}^{-1} {A_{2,-1}}^{-1}\right)$$

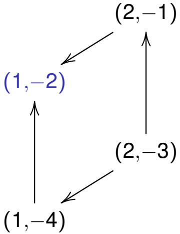
$$A_{1,r} = Y_{1,r-1} Y_{1,r+1} Y_{2,r}^{-1}, \quad A_{2,r} = Y_{2,r-1} Y_{2,r+1} Y_{1,r}^{-1}.$$

$K^{(1)}_{1,-2}$:



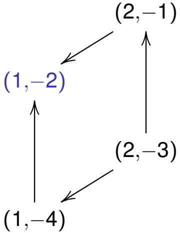
Type A_2

$K^{(1)}_{2,-2}$:

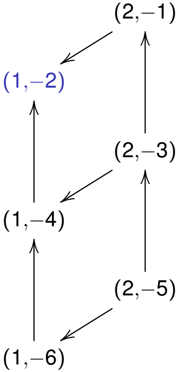


Type A_2

$K^{(1)}_{2,-2}$:

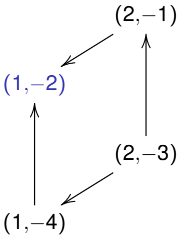


$K^{(1)}_{3,-2}$:

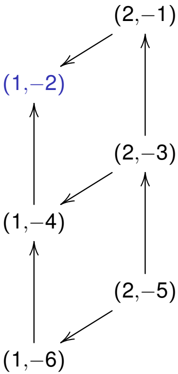


Type A_2

$K^{(1)}_{2,-2}$:



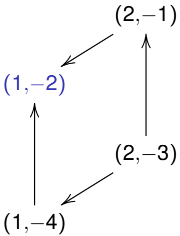
$K^{(1)}_{3,-2}$:



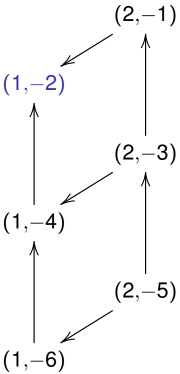
$\rightsquigarrow \quad \chi_q\left(W^{(1)}_{2,-5}\right)$

Type A_2

$K^{(1)}_{2,-2}$:



$K^{(1)}_{3,-2}$:

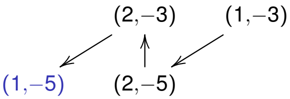


$\rightsquigarrow \quad \chi_q\left(W^{(1)}_{2,-5}\right)$

$\rightsquigarrow \quad \chi_q\left(W^{(1)}_{3,-7}\right)$

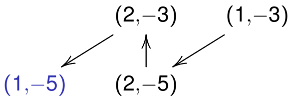
Type B_2

$K^{(1)}_{1,-5}$:



Type B_2

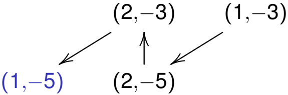
$K^{(1)}_{1,-5}$:



$\rightsquigarrow \quad \chi_q\left(W^{(1)}_{1,-7}\right)$

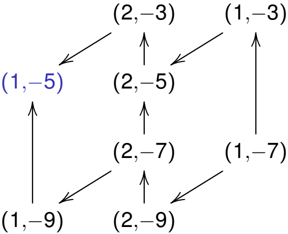
Type B_2

$K^{(1)}_{1,-5}$:



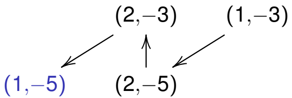
$\rightsquigarrow \quad \chi_q\left(W^{(1)}_{1,-7}\right)$

$K^{(1)}_{2,-5}$:



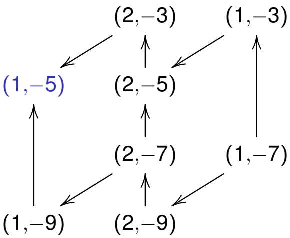
Type B_2

$K^{(1)}_{1,-5}$:



$\rightsquigarrow \quad \chi_q\left(W^{(1)}_{1,-7}\right)$

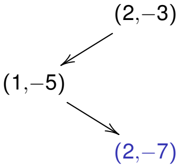
$K^{(1)}_{2,-5}$:



$\rightsquigarrow \quad \chi_q\left(W^{(1)}_{2,-11}\right)$

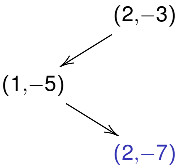
Type B_2

$K^{(2)}_{1,-7}$:



Type B_2

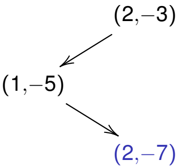
$K^{(2)}_{1,-7}$:



$\rightsquigarrow \quad \chi_q\left(W^{(2)}_{1,-8}\right)$

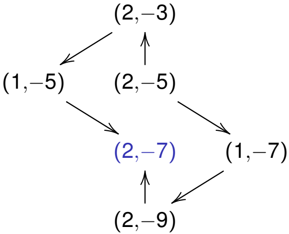
Type B_2

$K^{(2)}_{1,-7}$:



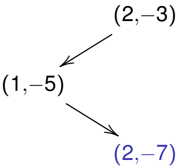
$\rightsquigarrow \quad \chi_q\left(W^{(2)}_{1,-8}\right)$

$K^{(2)}_{2,-7}$:



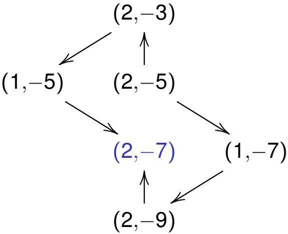
Type B_2

$K^{(2)}_{1,-7}$:



$\rightsquigarrow \quad \chi_q\left(W^{(2)}_{1,-8}\right)$

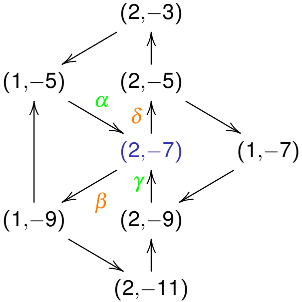
$K^{(2)}_{2,-7}$:



$\rightsquigarrow \quad \chi_q\left(W^{(2)}_{2,-10}\right)$

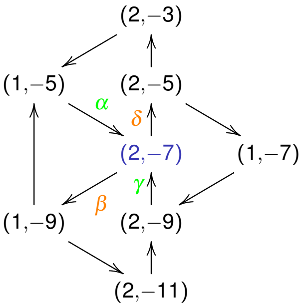
Type B_2

$K^{(2)}_{3,-7}$:



Type B_2

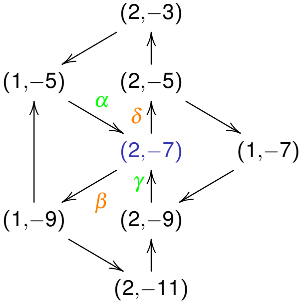
$K^{(2)}_{3,-7}$:



$$\alpha = \gamma = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \beta = \delta = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

Type B_2

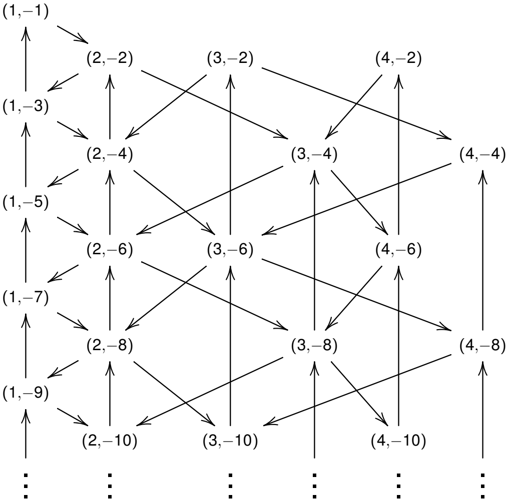
$K^{(2)}_{3,-7}$:



$$\alpha = \gamma = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \beta = \delta = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

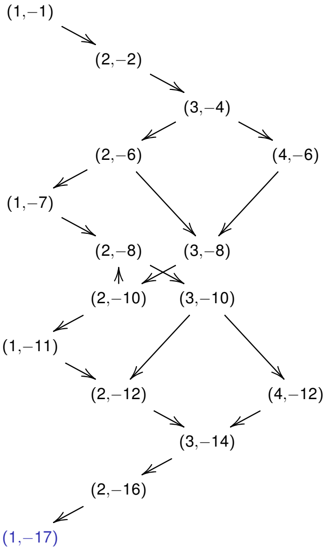
$$\rightsquigarrow \quad \chi_q\left(W^{(2)}_{3,-12}\right)$$

Type F_4: the quiver Q^-



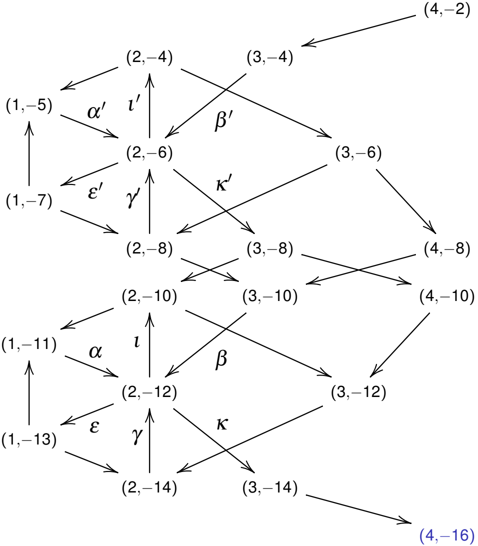
Type F_4

$K^{(1)}_{1,-17}$:



Type F_4

$K^{(4)}_{1,-16}$:



Relation with Nakajima theory

Relation with Nakajima theory

- If $z_i \rightsquigarrow M_i$ then $\prod_i z_i \rightsquigarrow \bigoplus_i M_i$

Relation with Nakajima theory

- If $z_i \rightsquigarrow M_i$ then $\prod_i z_i \rightsquigarrow \oplus_i M_i$
- $\rightsquigarrow$ geometric q-character formulas for tensor products of fundamental $U_q(\widehat{\mathfrak{g}})$-modules, that is, for standard modules.

Relation with Nakajima theory

- If $z_i \rightsquigarrow M_i$ then $\prod_i z_i \rightsquigarrow \oplus_i M_i$
- $\rightsquigarrow$ geometric q-character formulas for tensor products of fundamental $U_q(\widehat{\mathfrak{g}})$-modules, that is, for standard modules.
- In type A,D,E, the varieties of submodules involved in these formulas are isomorphic to Nakajima graded quiver varieties $\mathscr{L}^\bullet(V, W)$ (Lusztig, Savage-Tingley).

Relation with Nakajima theory

- If $z_i \rightsquigarrow M_i$ then $\prod_i z_i \rightsquigarrow \oplus_i M_i$
- $\rightsquigarrow$ geometric q-character formulas for tensor products of fundamental $U_q(\widehat{\mathfrak{g}})$-modules, that is, for standard modules.
- In type A,D,E, the varieties of submodules involved in these formulas are isomorphic to Nakajima graded quiver varieties $\mathscr{L}^\bullet(V, W)$ (Lusztig, Savage-Tingley).
- In type B,C,F,G, these varieties of submodules might be interesting replacements for the missing Nakajima varieties.

Relation with Nakajima theory

- If $z_i \rightsquigarrow M_i$ then $\prod_i z_i \rightsquigarrow \oplus_i M_i$
- $\rightsquigarrow$ geometric q-character formulas for tensor products of fundamental $U_q(\widehat{\mathfrak{g}})$-modules, that is, for standard modules.
- In type A,D,E, the varieties of submodules involved in these formulas are isomorphic to Nakajima graded quiver varieties $\mathscr{L}^\bullet(V, W)$ (Lusztig, Savage-Tingley).
- In type B,C,F,G, these varieties of submodules might be interesting replacements for the missing Nakajima varieties.
- A simple $U_q(\widehat{\mathfrak{g}})$-module S is called real if $S \otimes S$ is simple.

Relation with Nakajima theory

- If $z_i \rightsquigarrow M_i$ then $\prod_i z_i \rightsquigarrow \oplus_i M_i$
- $\rightsquigarrow$ geometric q-character formulas for tensor products of fundamental $U_q(\widehat{\mathfrak{g}})$-modules, that is, for standard modules.
- In type A,D,E, the varieties of submodules involved in these formulas are isomorphic to Nakajima graded quiver varieties $\mathscr{L}^\bullet(V, W)$ (Lusztig, Savage-Tingley).
- In type B,C,F,G, these varieties of submodules might be interesting replacements for the missing Nakajima varieties.
- A simple $U_q(\widehat{\mathfrak{g}})$-module S is called real if $S \otimes S$ is simple.

Conjecture

Real simple modules correspond to cluster monomials.

Relation with Nakajima theory

- If $z_i \rightsquigarrow M_i$ then $\prod_i z_i \rightsquigarrow \oplus_i M_i$
- $\rightsquigarrow$ geometric q-character formulas for tensor products of fundamental $U_q(\widehat{\mathfrak{g}})$-modules, that is, for standard modules.
- In type A,D,E, the varieties of submodules involved in these formulas are isomorphic to Nakajima graded quiver varieties $\mathscr{L}^\bullet(V, W)$ (Lusztig, Savage-Tingley).
- In type B,C,F,G, these varieties of submodules might be interesting replacements for the missing Nakajima varieties.
- A simple $U_q(\widehat{\mathfrak{g}})$-module S is called real if $S \otimes S$ is simple.

Conjecture

Real simple modules correspond to cluster monomials.

$\rightsquigarrow$ Would give geometric q-character formulas for all real simples.