T H I B O N
FEST

SLC 78

Δ

What kind of symmetric function is

Δ

?

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Δ

Definition(s)

The discriminant is an invariant

$$P(x) = x^n - e_1 x^{n-1} + e_2 x^{n-2} + \cdots + (-1)^{n-1} e_{n-1} x + (-1)^n e_n$$

$$\Delta(P) = \pm \begin{vmatrix} 1 & e_1 & e_2 & \cdots & e_{n-1} & e_n & 0 & \dots & 0 \\ 0 & 1 & e_1 & e_2 & \cdots & e_n & 0 & \dots & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & \cdots & 0 & 1 & e_1 & e_2 & \cdots & e_{n-1} & e_n \\ n & (n-1)e_1 & (n-2)e_2 & \cdots & e_{n-1} & e_n & 0 & \dots & 0 \\ 0 & 1 & e_1 & e_2 & \cdots & e_{n-1} & 0 & \dots & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & \cdots & 0 & n & (n-1)e_1 & (n-2)e_2 & \cdots & e_{n-1} & \end{vmatrix}$$

Δ is the resultant of P and P'.

$\Delta(P) \neq 0$ iff P has only simple roots.

An invariant

Determine if the Polynomial has only simple roots.

Δ

Discriminant as a symmetric function

$$\Delta(x_1, \ldots, x_n) = \pm \prod_{i \neq j} (x_i - x_j)^2.$$

In the Schur basis :

$\Delta(x_1, x_2) = -s_2 + 3\, s_{1,1}$

$\Delta(x_1, x_2, x_3) = -s_{4,2} + 3\, s_{4,1,1} + 3\, s_{3,3} - 6\, s_{3,2,1} + 15\, s_{2,2,2}$

$\Delta(x_1, x_2, x_3, x_4) =$ 16 terms,

$\Delta(x_1, x_2, x_3, x_4, x_5) =$ 59 terms,

$\Delta(x_1, x_2, x_3, x_4, x_5, x_6) =$ 247 terms,

$\Delta(x_1, x_2, x_3, x_4, x_5, x_6, x_7) =$ 1111 terms.

An invariant

Determine if the Polynomial has only simple roots.

Δ

A symmetric function

The square of the product of the differences of the variables.

Discriminant as the square of the Vandermonde determinant

$$\Delta(x_1, \ldots, x_n) = \det \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ & & \vdots & & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix}^2 .$$

An invariant

Determine if the Polynomial has only simple roots.

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A symmetric function

The square of the product of the differences of the variables.

The square of the Vandermonde determinant

The square of a determinant

An invariant

Determine if the Polynomial has only simple roots.

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Applications

A symmetric function

The square of the product of the differences of the variables.

The square of the Vandermonde determinant

The square of a determinant

Invariant Theory and its applications

- Involved in the classification of entanglement.
  From a proposal of Klyachko : use the (geometric) invariant theory to classify quantum systems of particles (qubit systems).
  8 papers with J.Y. Thibon and several co-authors.
- Used for computing hyperdeterminants

$$\mathrm{Det}(M_{i_1,\dots,i_{2k}})_{1\leq i_1,\dots,i_{2k}\leq N} = \frac{1}{N!}\sum_{\sigma_1,\dots,\sigma_{2k}\in\mathfrak{S}_N}\epsilon(\sigma_1)\cdots\epsilon(\sigma_{2k})\prod_{i=1}^{N} M_{\sigma_1(i)\dots\sigma_{2k}(i)}.$$

of Hankel type (i.e. $M_{i_1,\dots,i_{2k}} = f(i_1+\cdots+i_{2k})$).

An invariant

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Quantum Information (classification of entanglement) Hyperdeterminant

Invariant Theory

The square of the Vandermonde determinant

The square of a determinant

Random matrices

The joint probability density for the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of **GUE**/**GOE**/**GSE** (**G**aussian **U**nitary/**O**rthogonal/**S**ymplectic) is given by

$$P_{N,\beta}(\lambda_1, \ldots, \lambda_N) = C_{N,\beta} \exp\left(\frac{-\sum_{i=1}^{N} \lambda_i^2}{2}\right) \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} \prod d\lambda_i$$

where $\beta = 1(U), 2(O), 4(S)$.
Selberg integral :

$$\begin{aligned} S(N;\alpha,\beta;\gamma) &:= \int_0^1 \prod_{i<j} |\lambda_i - \lambda_j||^{2\gamma} \prod_{j=1}^{N} \lambda_j^{\alpha-1}(1-\lambda_j)^{\beta-1} d\lambda_j \\ &= \prod_{j=0}^{N-1} \frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)}{\Gamma(1+\gamma)\Gamma(\alpha+\beta+(n+j-1)\gamma)} \end{aligned}$$

Selberg proof : for $\gamma \in \mathbb{N}$ and extended to $\gamma \in \mathbb{C}$ using analytic tools (Carlson Theorem).

Random matrices

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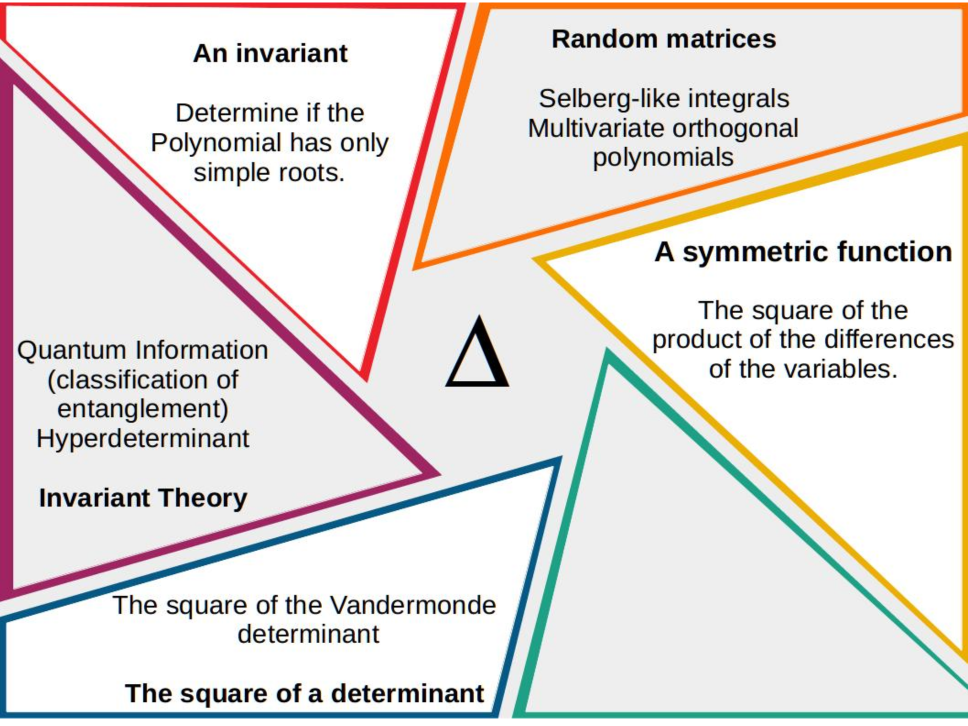
where $\beta = 1(U), 2(O), 4(S)$.
Selberg-like integrals :

$$\int \ldots \int f(\lambda_1, \ldots, \lambda_{N)} \prod_{i<j} |\lambda_i - \lambda_j|^{2\beta} d\mu(\lambda_1) \ldots d\mu(\lambda_N) = ??$$


An invariant
Determine if the Polynomial has only simple roots.
Random matrices
Selberg-like integrals
Multivariate orthogonal polynomials
A symmetric function
The square of the product of the differences of the variables.
Quantum Information (classification of entanglement)
Hyperdeterminant
Invariant Theory
The square of the Vandermonde determinant
The square of a determinant
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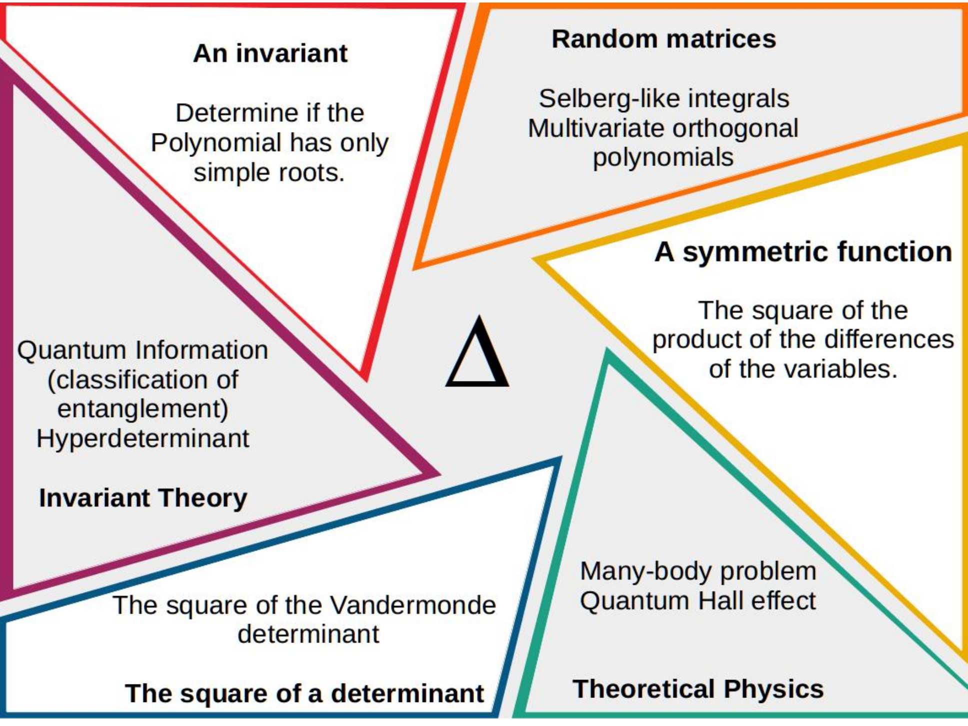
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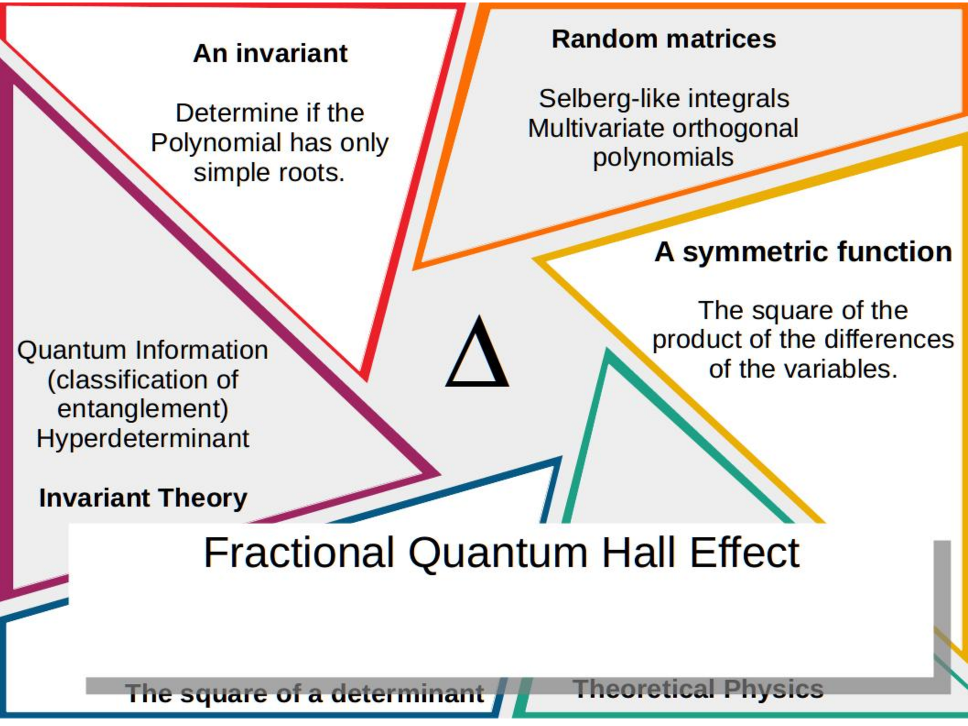
Many-body problem
Quantum Hall effect
Theoretical Physics
The square of the Vandermonde determinant
The square of a determinant









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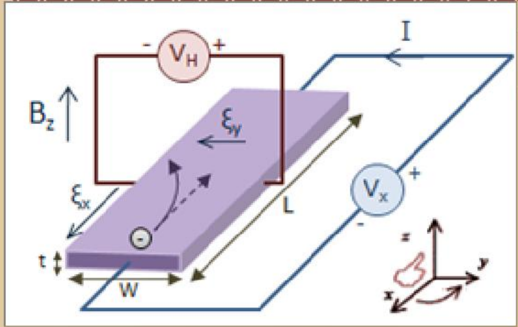
Invariant Theory
Fractional Quantum Hall Effect
The square of a determinant
Theoretical Physics

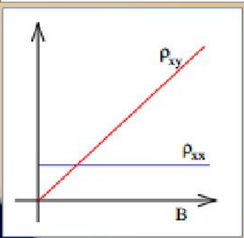
(Classical) Hall Effect



J : Current density
E: Electric Field
σ: matrix relating J and E (Ohm law)

$J = \sigma E$

conductivity

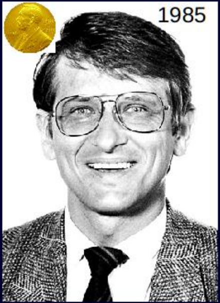


Resistivity / Magnetic field(B)

$$\rho = \sigma^{-1} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{yy} \end{pmatrix}$$

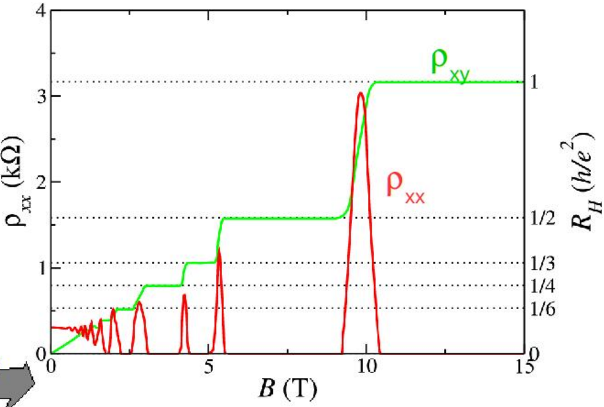
Resitivity

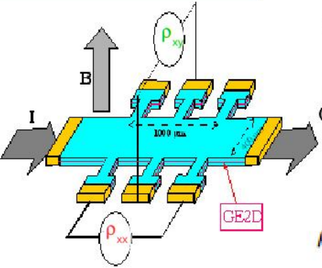
Edwin Herbert Hall
(1855-1938)



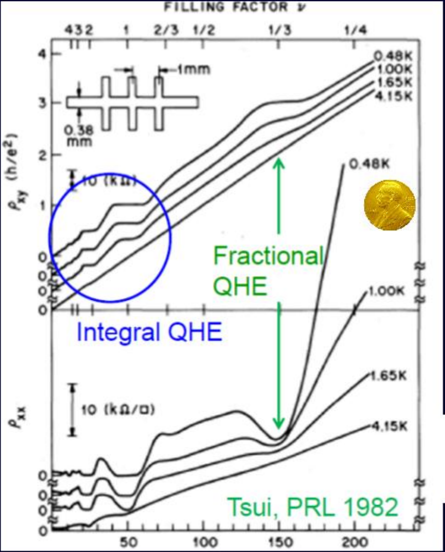
Klaus von Klitzing

Integer Quantum Hall Effect





$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu} \quad \nu \in \mathbf{Z}$$



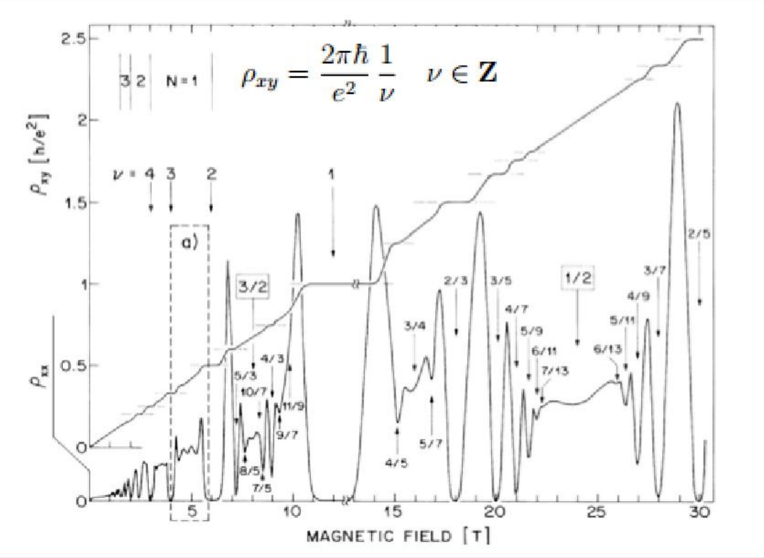
H. L. Strömer R.B. Laughlin D.C. Tsui

$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu} \quad \nu \in \mathbf{Q}$$

As the disorder is decreased, the integer Hall plateaux become less prominent. But other plateaux emerge at fractional values.

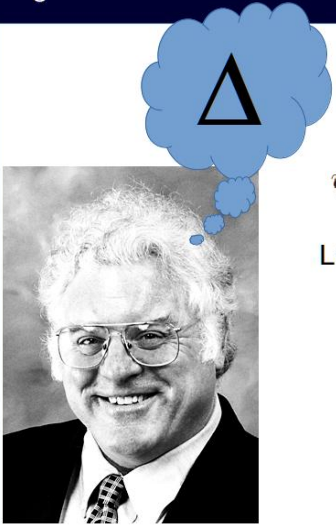
Fractional Quantum Hall Effect

Fractional Quantum Hall Effect



As the disorder is decreased, the integer Hall plateaux become less prominent. But other plateaux emerge at fractional values.

The comprehension of the phenomenom involves the many-body problem of electrons in Coulombien interaction in a bi-dimensional gaz.



Wavefunction under the form

$$\psi(z_1, \ldots, z_n) = f(z_1, \ldots, z_N) e^{-\sum_{i=1}^{N} |z_i|^2 / 4 l_B^2}$$

Laughlin : At $\nu = \frac{1}{m}$ (*m* odd)

$$\prod_{i<j} (z_i - z_j)^m e^{-\sum_{i=1}^{n} |z_i|^2 / 4 l_B^2}$$

$$f(z) = \Delta^{(m-1)/2} V(z)$$

A first question

Physicists want to write the Laughlin wave function in terms of Slater wave functions (wave functions due to John C. Slater of a multi-fermionic system, 1929).
Combinatorially : expand the powers of the discriminant on Schur functions.
Computation : usually by numerical methods.
No combinatorial interpretation.

- P. Di Francesco, M. Gaudin, C. Itzykson, F. Lesage, *Laughlin's wave functions, Coulomb gases and expansions of the discriminant*, Int.J.Mod.Phys. (1994)
- T. Scharf, J.-Y. Thibon, B.G. Wybourne, *Powers of the Vandermonde determinant and the quantum Hall effect*, J. Phys.(1994)
- R.C. King, F. Toumazet, B.G. Wybourne, *The square of the Vandermonde determinant and its q-generalization*, J. Phys. A (2004)

Second question

What about the other values $\nu \in \mathbb{Z}$?

$\nu = \frac{1}{2m}$	$\nu = \frac{1}{2m+1}$	$\nu = \frac{p}{2pm+2}$
Laughlin	Moore-Read	Read-Rezayi
$\prod_{i<j}(z_i - z_j)^m$	$Pf\left(\frac{1}{z_i - z_j}\right) \times \prod_{i<j}(z_i - z_j)^{2m-1}$	$\mathcal{S}\left(\prod_{k=1}^{p} \prod_{(k-1)\frac{N}{p}<i<j\leq k\frac{N}{p}} (z_i - z_j)^2\right) \times \prod_{i<j}(z_i - z_j)^{2m}$

Bernevig-Haldane : General expression for $\nu = \frac{k}{r}$ in terms of Jack polynomials :

$$J^{-\frac{k+1}{r-1}}_{((p-1)r)^k((p-2)r)^k\cdots(r)^k0^k}(z_1, \ldots, z_{pk}).$$

Why Jack polynomials ?

Not the solutions of the true eigenvector problem but adiabatically equivalent.
Conditions that the wave function must fulfill.

- Eigenfunction of Laplace-Beltrami type operator with dominance properties ($\alpha = -\frac{k+1}{r-1}$) :

$$\mathcal{H}_{LB}^{(\alpha)} = \sum_{i=1}^{N} \left(z_i \frac{\partial}{\partial z_i} \right)^2 + \frac{1}{\alpha} \sum_{i<j} \frac{z_i + z_j}{z_i - z_j} \left(z_i \frac{\partial}{\partial z_i} - z_i \frac{\partial}{\partial z_j} \right)$$

  (Jack polynomials)
- In the kernel of $L^+ = \sum_i \frac{\partial}{\partial z_i}$. Invariant by translation ; highest weighted ;singular.
- Eigenfunctions of of $L^0 = \sum_i z_i \frac{\partial}{\partial z_i}$. Homogeneous.
- In the kernel of $L^- = \sum_i z_i^2 \frac{\partial}{\partial z_i}$. Lowest weight.
- Clustering conditions :

Why Jack polynomials ?

Clustering conditions at $\nu = \frac{k}{r}$

k particles cluster.
Setting $z_1 = \cdots = z_k$ the wave function must vanish as

$$\prod_{i=k+1}^{N} (z_1 - z_i)^r.$$

Related to Feigin et al (math.QA/0112127). Wheel conditions.

What are Jack polynomials ?

Hecke algebra

Action on multivariate polynomials :

$$T_i = t + (s_i - 1)\frac{tx_{i+1} - x_i}{x_{i+1} - x_i}.$$

In particular $1\,T_i = 1$ and $x_{i+1}\,T_i = x_i$.
Together with the multiplication by the variables x_i and the affine operator τ defined by

$$\tau f(x_1, \ldots, x_N) = f(\frac{x_N}{q}, x_1, \ldots, x_{N-1}).$$

We have

- $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ (braid relation)
- $T_i T_j = T_j T_i$ for $|i - j| > 1$
- $(T_i - t)(T_i + 1) = 0$

What are Jack polynomials ?

Cherednik operators and Macdonald polynomials

Cherednik operators :

$$\xi_i = t^{1-i} T_{i-1} \cdots T_1 \tau T_{N-1}^{-1} \cdots T_i^{-1}$$

Knop-Cherednik operators :

$$\Xi_i = t^{1-i} T_{i-1} \cdots T_1 (\tau - 1) T_{N-1}^{-1} \cdots T_i^{-1}$$

Non symmetric Macdonald polynomials : $E_v = (*) x_1^{v[1]} \cdots x_N^{v[N]} + \cdots$ simultaneous eigenfunctions of ξ_i.

Non symmetric shifted Macdonald polynomials : $M_v = (*) x_1^{v[1]} \cdots x_N^{v[N]} + \cdots$ simultaneous eigenfunctions of Ξ_i.

Remark :

$$M_v = E_v + \sum_{|u|<|v|} \alpha_u E_u.$$

What are Jack polynomials ?

Spectral vectors and vanishing properties

$v = [0, 1, 2, 2, 0, 1]$: Standardized :

$$\mathrm{std}_v = [1, 3, 5, 4, 0, 2]$$

Spectral vector : $\mathrm{Spec}_{v[i]} = \frac{1}{\langle v\rangle[i]}$ with

$$\langle v\rangle = [t^1, qt^3, q^2t^5, q^2t^4, 1, qt^2].$$

Shifted Macdonald polynomials can alternatively be defined by vanishing properties

$$M_v(\langle u\rangle[1], \ldots, \langle u\rangle[N]) = 0$$

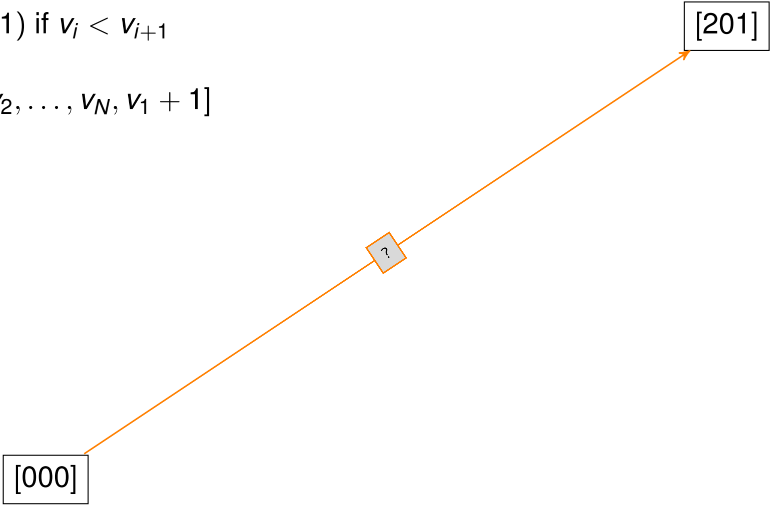
for $|u| \leq |v|$ and $u \neq v$.

What are Jack polynomials ?

Yang-Baxter graph

$(i, i+1)$ if $v_i < v_{i+1}$

$$v \cdot \Phi = [v_2, \dots, v_N, v_1 + 1]$$



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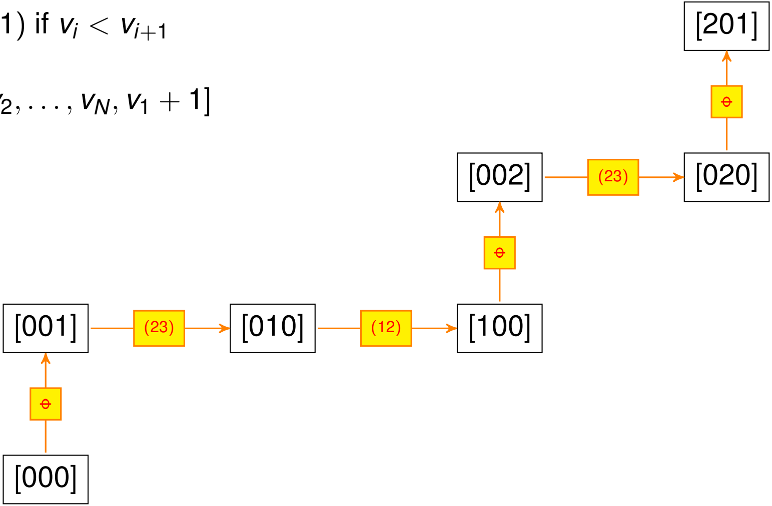
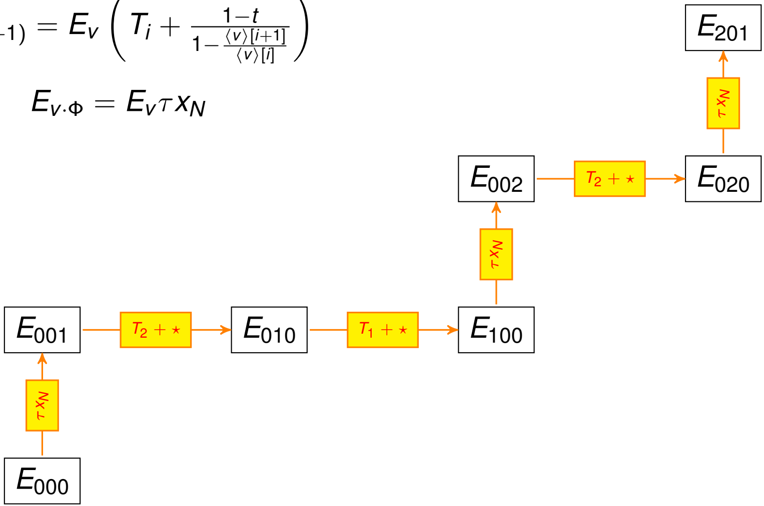


What are Jack polynomials ?

Yang-Baxter graph

$$E_{v\cdot(i,i+1)} = E_v \left(T_i + \frac{1-t}{1-\frac{\langle v\rangle[i+1]}{\langle v\rangle[i]}} \right)$$

$$E_{v\cdot\Phi} = E_v \tau x_N$$



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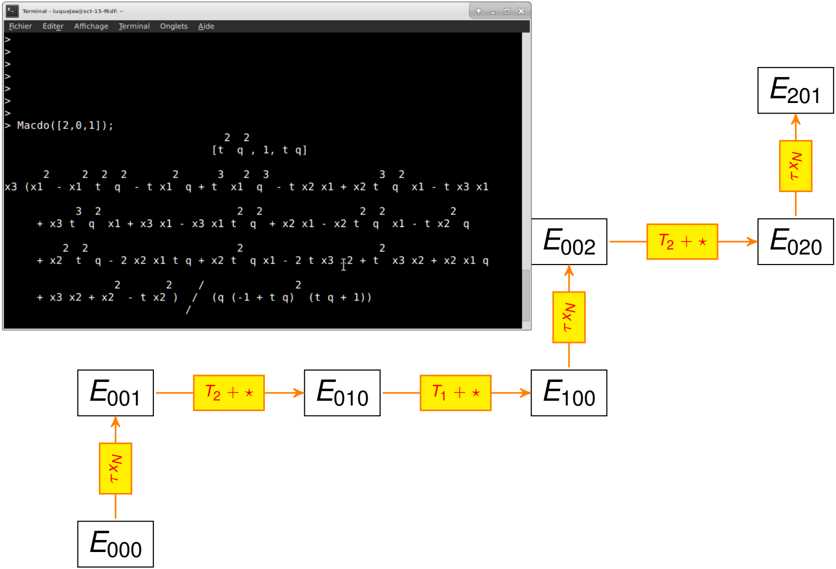

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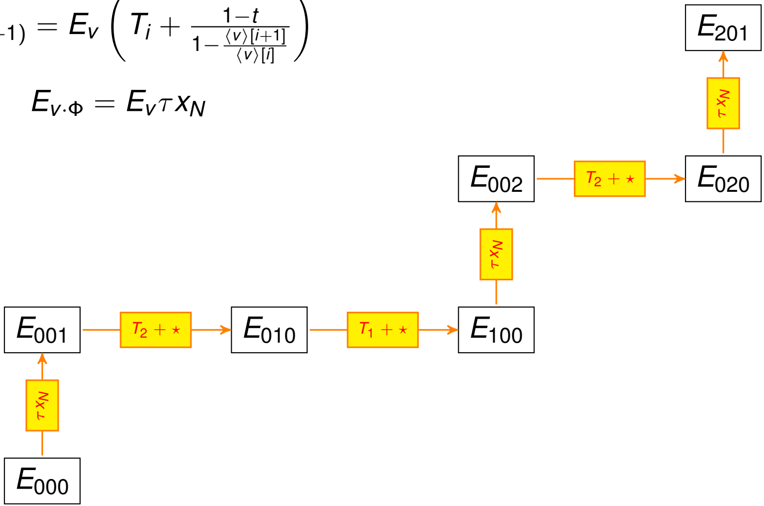

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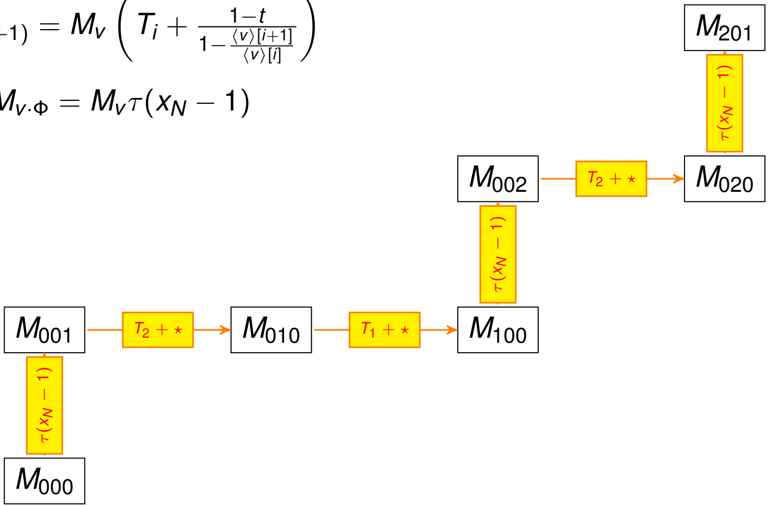
$$M_{v\cdot\Phi} = M_v \tau(x_N - 1)$$



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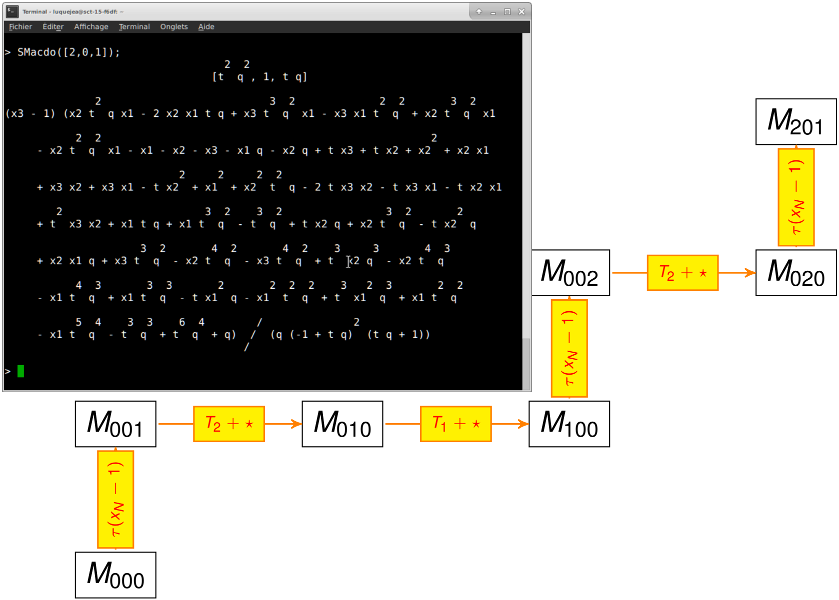


What are Jack polynomials ?

Symmetrization

Symmetrizing operator :

$$\mathcal{S}_N = \sum_{s \in \mathfrak{S}_N} T_\sigma$$

where $T_\sigma = T_{i_1} \dots T_{i_k}$ if $\sigma = s_{i_1} \cdots s_{i_k}$ is the shortest decomposition of σ in elementary transpositions.

Symmetric Macdonald polynomials : eigenfunctions of symmetric polynomials in the variables ξ_i.

$$J_\lambda = (*) E_\lambda . \mathcal{S}_N .$$

Symmetric shifted Macdonald polynomials : eigenfunctions of symmetric polynomials in the variables Ξ_i.

$$MS_\lambda = (*) M_\lambda . \mathcal{S}_N .$$

What are Jack polynomials ?

Symmetric Macdonald polynomials

Relationship :

$$MS_\lambda = (*)J_\lambda + \sum_{\mu \subsetneq \lambda} \alpha_\mu J_\mu$$

Symmetric Shifted Macdonald polynomials are defined by interpolation :

$$MS_\lambda(\langle\mu\rangle) = 0 \text{ for } |\lambda| \leq |\mu| \text{ and } \lambda \neq \mu.$$

Sekiguchi-Debiard operators : J_λ and MS_λ are eigenfunctions of (resp.) $\xi = \sum \xi_i$ and $\Xi = \sum \Xi_i$ with eigenvalues

$$q^{\lambda_1} t^{N-1} + \cdots q^{\lambda[N-1]} t^1 + q^{\lambda[N]}.$$

What are Jack polynomials ?

From Macdonald polynomials to Jack polynomials

Four types of Macdonald polynomials

	Symmetric	*NonSymmetric*
Homogeneous	J	E
Shifted	MS	M

Jack polynomials : $q = t^{\alpha}$ and $t \to 1$.

Four types of Jack polynomials

	Symmetric	*NonSymmetric*
Homogeneous	J	E
Shifted	MS	M

Dunkl operator and singular polynomials

Definition : $D_i = \Xi_i - \xi_i$.
Singular polynomial = in the kernel of each D_i.
Remark if E_v is singular and the eigenspace has dimension 1 then $E_v = M_v$.
There exist singular Macdonald polynomials for some specializations of the parameters $q^* t^\star = 1$.

Application : singular E_v are defined by vanishing properties.

$$L^+_{q,t} := \frac{1}{1-q} \sum D_i, \ \lim_{t \to 1} L^+_{q,t} = L^+ = \sum_i \frac{\partial}{\partial z_i}$$

Clustering properties and factorization

Aim : deduce clustering properties of Jack polynomials from factorization of Macdonald polynomials.

Example 1 :

$$J_{420}(x_1, x_2, x_3 : q = t^{-2}, t) = (*) \prod_{i \neq j} (x_i - t x_j)$$

$$J^{(-2)}_{420}(x_1, x_2, x_3) = (*)(x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3)^2$$

Discriminant=Jack=Laughlin state !

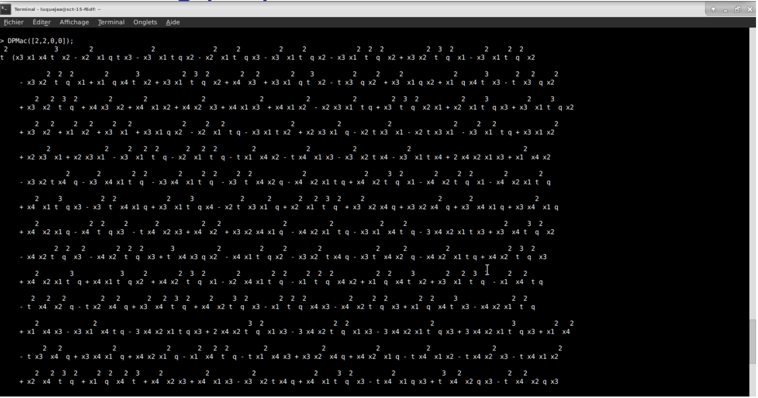
Example 2 :

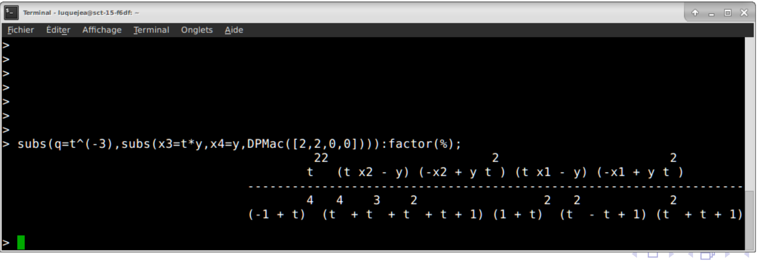
$$J_{[2,2,0,0]}(x_1, x_2, ty, y; q = t^{-3}, t) = (*)(x_1 - t^{-1} y)(x_1 - t^2 y)(x_2 - t^{-1} y)(x_2 - t^2 y)$$

$$J^{(-3)}_{[2,2,0,0]}(x_1, x_2, y, y) = (*)(x_1 - y)^2 (x_2 - y)^2.$$

Clustering properties $r = 2, k = 2$.(More-Read state).

Clustering properties and factorization





An example

2 Clusters of order 2 with 6 particles.

Assume $qt^3 = 1$.
Feigin et al : P satisfies the wheel conditions if $\frac{x_2}{x_1}, \frac{x_3}{x_2}, \frac{x_1}{x_3} \in \{t, tq\}$ implies $P(x_1, \ldots, x_6) = 0$
Let $J_6^{2,2}$ be the ideal of symmetric polynomials satisfying the wheel conditions.
Two results by Feigin et al. :

- $J_6^{2,2}$ is spanned by Macdonald J_λ such that $\lambda_i - \lambda_{i+2} \geq 2$ for $1 \leq i \leq 4$.
- $J_6^{2,2}$ is stable by $L^+_{q,t}$.

The minimal degree polynomial belonging to $J_6^{2,2}$ is J_{442200}. So $J_{442200} \in \mathrm{ker} L^+_{q,t}$.
J_{442200} has the same eigenvalues as MS_{442200} for Ξ.

An example

Examination of the eigenvalues of Ξ

$\mu \subset 442200$	$\langle \lambda \rangle$
4422	$q^4t^5 + q^4t^4 + q^2t^3 + q^2t^2 + t + 1$
4421	$q^4t^5 + q^4t^4 + q^2t^3 + qt^2 + t + 1$
4420	$q^4t^5 + q^4t^4 + q^2t^3 + t^2 + t + 1$
4411	$q^4t^5 + q^4t^4 + qt^3 + qt^2 + t + 1$
4410	$q^4t^5 + q^4t^4 + qt^3 + t^2 + t + 1$
$\vdots$	
1000	$qt^5 + t^4 + t^3 + t^2 + t + 1$
0000	$t^5 + t^4 + t^3 + t^2 + t + 1$

An example

Examination of the eigenvalues of Ξ ($q = t^{-3}$)

$\mu \subset 442200$	$\langle\lambda\rangle$
4422	$t^{-7} + t^{-8} + t^{-3} + t^{-4} + t + 1$ ← the only one !
4421	$t^{-7} + t^{-8} + t^{-3} + t^{-1} + t + 1$
4420	$t^{-7} + t^{-8} + t^{-3} + t^{2} + t + 1$
4411	$t^{-7} + t^{-8} + 1 + t^{-1} + t + 1$
4410	$t^{-7} + t^{-8} + 1 + t^{2} + t + 1$
$\vdots$	
1000	$t^{2} + t^{4} + t^{3} + t^{2} + t + 1$
0000	$t^{5} + t^{4} + t^{3} + t^{2} + t + 1$

So $J_{442200} = (*)MS_{442200}$.

An example

On MS_{442200}

- For $N = 6, q = t^{-3}$ it is homogeneous !

$$MS_{442200}(x_1, x_2, x_3, x_4, ty, y) = y^{12} MS_{442200}\left(\frac{x_1}{y}, \frac{x_2}{y}, \frac{x_3}{y}, \frac{x_4}{y}, t, 1\right).$$

An example

On MS_{442200}

- For $N = 6, q = t^{-3}$ it is homogeneous !

$$MS_{442200}(x_1, x_2, x_3, x_4, ty, y) = y^{12} MS_{442200}\left(\frac{x_1}{y}, \frac{x_2}{y}, \frac{x_3}{y}, \frac{x_4}{y}, t, 1\right).$$

- Vanishing properties of $MS_{442200}\left(x_1 t^2, x_2 t^2, x_3 t^2, x_4 t^{-2}, t, 1\right)$ and $MS_{4422}(x_1, x_2, x_3, x_4)$ are the same.

MS_{442200}		MS_{4422}	
551100	$[q^5 t^5, q^5 t^4, qt^3, qt^2, t, 1]$	5511	$[q^5 t^3, q^5 t^2, qt, q]$
442100	$[q^4 t^5, q^4 t^4, q^2 t^3, qt^2, t, 1]$	4421	$[q^4 t^3, q^4 t^2, q^2 t, q]$
442000	$[q^4 t^5, q^4 t^4, q^2 t^3, t^2, t, 1]$	4420	$[q^4 t^3, q^4 t^2, q^2 t, 1]$
441100	$[q^4 t^5, q^4 t^4, qt^3, qt^2, t, 1]$	4411	$[q^4 t^3, q^4 t^2, qt, q]$
etc.			

An example

On MS_{442200}

- For $N=6, q=t^{-3}$ it is homogeneous !

$$MS_{442200}(x_1,x_2,x_3,x_4,ty,y)=y^{12}MS_{442200}\left(\frac{x_1}{y},\frac{x_2}{y},\frac{x_3}{y},\frac{x_4}{y},t,1\right).$$

- Vanishing properties of $MS_{442200}\left(x_1t^2,x_2t^2,x_3t^2,x_4t^2,t,1\right)$ and $MS_{4422}(x_1,x_2,x_3,x_4)$ are the same.
- $P(x_1,x_2,x_3,x_4)=$ $\prod_{i=1}^4(x_i-1)(x_i-q)MS_{2200}(q^{-2}x_1,q^{-2}x_2,q^{-2}x_3,q^{-2}x_4)$ has the same dominant monomial and the same vanishing properties as MS_{4422}.
  P vanishes for $\langle abc0\rangle$ ($x_4=1$) and $\langle abc1\rangle$ ($x_4=q$).
  Now let $\mu=[a+2,b+2,c+2,d+2]$ with $\mu\vdash n\leq 12$ with $\mu\neq[4,4,2,2]$ then MS_{2200} vanishes for $\langle[a,b,c,d]\rangle$ ($[a,b,c,d]\vdash n-8\leq 4$).

An example

Final computation

$$J_{442200}(x_1,\ldots,x_4,ty,y)=(*)\prod_{i=1}^{4}(x_i-yt^2)(x_i-yqt^2)MS_{2200}\left(\frac{x_1}{yq^2t^2},\ldots,\frac{x_4}{yq^2t^2}\right)$$

Similar reasoning : MS_{2200} is homogeneous for $q=t^{-3}$.
Examination of vanishing properties shows

$$MS_{2200}(x_1,x_2,ty,y)=(*)(x_1-yt^2)(x_1-yqt^2)(x_2-yt^2)(x_2-yqt^2).$$

Finally,

$$J_{442200}(x_1,x_2,ty_1,y_1,ty_2,y_2)=(*)\prod_{i=1}^{4}\prod_{j=1}^{2}(x_i-t^2y_j)(x_i-qt^2y_j).$$

$t\to 1$:

$$J^{(-3)}_{442200}(x_1,x_2,y_1,y_1,y_2,y_2)=(*)\prod_{i=1}^{4}\prod_{j=1}^{2}(x_i-y_j)^2.$$

Read-Rezayi state.

Other factorizations

Not considered by Bernevig and Haldane.
Physical interpretation ?

$$P_{630}(x_1, x_2, x_3; q = -t^{-1}, t) = (-)_t(x_1+x_2)(x_1+x_3)(x_2+x_3)(x_1-tx_2)(x_1-tx_3)(x_2-tx_1)(x_2-tx_3)(x_3-tx_1)(x_3-tx_1),$$

$$P_{53000}(x_1, x_2, y, yt, yt^2; q = t^{-2}, t) = (-)_t(x_1-yt^3)(x_1-yt)(x_1-yt^{-1})(x_2-yt^3)(x_2-yt)(x_2-yt^{-1}),$$

$$P_{42200}(x_1, y_1, y_1t, y_2, y_2t; q = t^{-3}, t) = (-)_t(x_1-y_1t^2)(x_1-y_2t^2)(x_1-y_1t)(x_1-y_2t)(y_1-ty_2)(y_1-t^2y_2)(y_2-ty_1)(y_2-t^2y_1),$$

$$P_{6400000}(x_1, y_1, y_1t, y_1t^2, y_2, y_2t, y_2t^2; q = t^{-2}, t) = (-)_t(x_1-y_1t^3)(x_1-y_1t^3)(x_1-y_1t^{-1})(x_1-y_2t^3)(x_1-y_2t^3)(x_1-y_2t^{-1})(y_1-ty_2)(y_1-t^3y_2)(y_2-ty_1)(y_2-t^3y_1),$$

$$P_{7507}(x_1, y_1, \ldots, y_1t^4, y_2, \ldots, y_2t^2; q = t^2, t) = (-)_t(y_1-x_1t^3)(y_1-y_2t^3)(y_1-x_1t^5)(y_1-y_2t^5)(y_2-x_1t^3)(y_2-y_1t^3)P_{420}(x_1, y_1, y_2; q = t^{-2}, t), \ldots$$

Can not be obtained using wheel conditions but by proving they belongs in Ker $L^{+}_{q,t}$ [L Jolicoeur]

Other factorizations

Non symmetric Macdonald polynomials

$$E_{210}(x_1,x_2,x_3;q=z^{-2},t=z)=(-)_t(tx_2-x_1)(tx_3-x_1)(tx_3-x_2)$$

$$E_{630}(x_1,x_2,x_3;q=z^{-2},t=z^3)=(-)_z\,(x_2\,z-x_3)\,(-zx_3+x_2)\,(x_2-z^3x_3)\,(x_1\,z-x_3)$$
$$(-zx_3+x_1)\,(x_1-z^3x_3)\,(x_1\,z-x_2)\,(x_1-x_2\,z)\,(x_1-x_2\,z^3)\,,$$

$$E_{420}(x_1,x_2,x_3;q=-t^{-1},t)=(-)_t(x_2+x_3)(-tx_3+x_2)(x_3+x_1)(-tx_3+x_1)(x_1+x_2)(x_1-x_2t),$$

$$E_{221100}(x_1,x_2,y_1,ty_1,y_2,ty_2;q=t^{-3},t)=(-)_t(y_1-y_2t^2)(y_1-y_2t)(x_2-y_2t^2)$$
$$(x_2-t^2y_1)(x_1-y_2t^2)(x_1-t^2y_1),$$

$$E_{442200}(x_1,y_1,y_1z^2,y_2,y_2z^2;q=z^{-3},t=z^2)=(-)_z(y_1-y_2z^4)(y_1-y_2z)(y_1z-y_2)$$
$$(y_1-y_2z^2)(x_1-y_2z^4)(x_1-y_2z)(x_1-y_1z^4)(x_1-y_1z).$$

Joint work (in progress) with C. Dunkl and L. Colmenarejo

THANK YOU