Hook formulas for skew shapes

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Standard Young Tableaux

Irreducible representations of S_n:

Specht modules $\mathbb{S}_\lambda$, for all $\lambda \vdash n$.

Basis for $\mathbb{S}_\lambda$: **Standard Young Tableaux** of shape λ:
$\lambda = (2, 2, 1)$:

1 2		1 2		1 3		1 3		1 4
3 4		3 5		2 4		2 5		2 5
5		4		5		4		3

Hook-length formula [Frame-Robinson-Thrall]:

$$\dim \mathbb{S}_\lambda = \#\{\text{SYTs of shape } \lambda\} = f^\lambda = \frac{|\lambda|!}{\prod_{u \in \lambda} h_u} = \frac{5!}{4 * 3 * 2 * 1 * 1}$$

Hook length of box $u = (i, j) \in \lambda$: $h_u = \lambda_i - j + \lambda'_j - i + 1 = \#\left\{ \blacksquare \in \right\}$
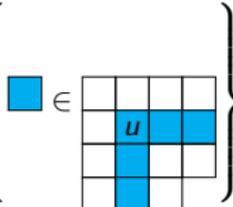
Counting skew SYTs

Outer shape λ, inner shape μ, e.g. for $\lambda = (5,4,4,2), \mu = (3,2,1)$
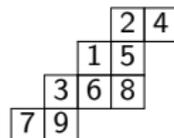
Jacobi-Trudi[Feit 1953]:

$$f^{\lambda/\mu} = |\lambda/\mu|! \cdot \det \left[\frac{1}{(\lambda_i - \mu_j - i + j)!} \right]_{i,j=1}^{\ell(\lambda)}.$$

Counting skew SYTs
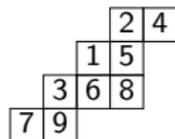
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Littlewood-Richardson:

$$f^{\lambda/\mu} = \sum_{\nu} c_{\mu,\nu}^{\lambda} f^{\nu}$$

3

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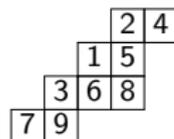
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Littlewood-Richardson:

$$f^{\lambda/\mu} = \sum_\nu c^\lambda_{\mu,\nu} f^\nu$$

No product formula, e.g. $\lambda/\mu = \delta_{n+2}/\delta_n$:

$f^{\delta_{n+2}/\delta_n} = E_{2n+1}$:
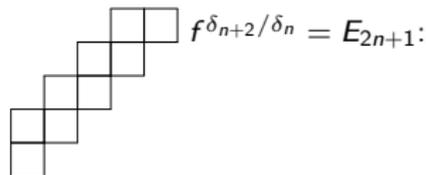
$$1 + E_1 x + E_2 \frac{x^2}{2!} + E_3 \frac{x^3}{3!} + E_4 \frac{x^4}{4!} + \ldots = \sec(x) + \tan(x).$$

Euler numbers: $2, 5, 16, 61 \ldots$

Hook-Length formula for skew shapes

Theorem (Naruse, SLC, September 2014)

$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in [\lambda] \setminus D} \frac{1}{h(u)},$$

where $\mathcal{E}(\lambda/\mu)$ is the set of excited diagrams of λ/μ.

Excited diagrams:

$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda : \text{ obtained from } \mu \text{ via } \;\blacksquare\; \longrightarrow \;\square\; \}$
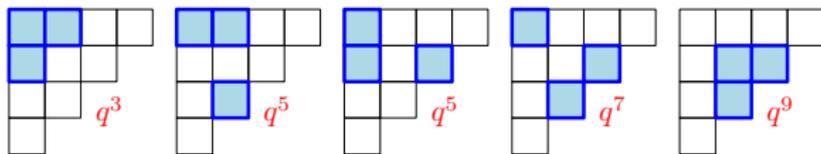
$$f^{(4321/21)} = 7! \left(\frac{1}{1^4 \cdot 3^3} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^2 \cdot 3^3 \cdot 5^2} + \frac{1}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7} \right) = 61$$

Hook-Length formula for skew shapes
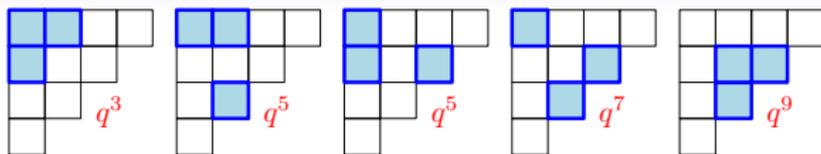
$$s_{\lambda/\mu}(1, q, q^2, \ldots) = \sum_{T \in SSYT(4321/21)} q^{|T|} = \frac{q^3}{(1-q)^4(1-q^3)^3} + 2 \times \frac{q^5}{(1-q)^3(1-q^3)^3(1-q^5)}$$

$$+ \frac{q^7}{(1-q)^2(1-q^3)^3(1-q^5)^2} + \frac{q^9}{(1-q)^2(1-q^3)^2(1-q^5)^2(1-q^7)}$$

Theorem (Morales-Pak-P)

$$\sum_{T \in SSYT(\lambda/\mu)} q^{|T|} = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in [\lambda] \setminus D} \left[\frac{q^{\lambda'_j - i}}{1 - q^{h(i,j)}} \right].$$

Theorem (Morales-Pak-P)

$$\sum_{\pi \in RPP(\lambda/\mu)} q^{|\pi|} = \sum_{S \in PD(\lambda/\mu)} \prod_{u \in S} \left[\frac{q^{h(u)}}{1 - q^{h(u)}} \right].$$

where $PD(\lambda/\mu)$ is the set of pleasant diagrams.
Other recent proof by [M. Konvalinka]

Algebraic proof for SSYTs:

$v = 245613$, $w = 361245$

[Ikeda-Naruse, Kreiman]: Let $w \preceq v$ be Grassmannian permutations whose unique descent is at position d with corresponding partitions $\mu \subseteq \lambda \subseteq d \times (n - d)$. Then

$$[X_w]\big|_v = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in D} \big(y_{v(d+j)} - y_{v(d-i+1)}\big).$$

Algebraic proof for SSYTs:
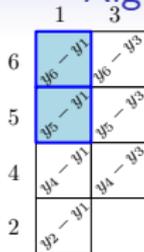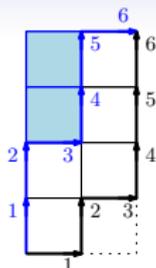
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Factorial Schur functions:

$$s_\mu^{(d)}(\mathbf{x}|\mathbf{a}) := \frac{\det\left[(x_j - a_1) \cdots (x_j - a_{\mu_i + d - i}) \right]_{i,j=1}^d}{\prod_{1 \le i < j \le d} (x_i - x_j)},$$

[Knutson-Tao, Lakshmibai–Raghavan–Sankaran] Schubert class at a point:

$$[X_w]\big|_v = (-1)^{\ell(w)} s_\mu^{(d)}\left(y_{v(1)}, \ldots, y_{v(d)} | y_1, \ldots, y_{n-1} \right).$$

5

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Evaluation at $y = 1, q, q^2, \ldots$, $v(d+1-i) = \lambda_i + d + 1 - i$, $x_i \to y_{v(i)} = q^{\lambda_i + d + 1 - i}$
$\to$ Jacobi-Trudi

$$s_\mu^{(d)}(q^{v(1)}, \ldots | 1, q, \ldots) = \frac{\det\left[\prod_{r=1}^{\mu_j + d - j} (q^{\lambda_i + d + 1 - i} - q^r) \right]_{i,j=1}^d}{\prod_{i<j} (q^{\lambda + d + 1 - i} - q^{\lambda_j + d + 1 - j})} = \ldots.$$

$$\ldots[simplifications]\ldots = \det[h_{\lambda_i - i - \mu_j + j}(1, q, \ldots)] = s_{\lambda/\mu}(1, q, \ldots)$$

Combinatorial proofs:

Hillman-Grassl algorithm/map Φ: Reverse Plane Partitions of shape λ to Arrays of shape λ:

$$RRP \quad P = \begin{array}{|c|c|c|}\hline 0&1&2\\\hline 1&1&3\\\hline 2\\\cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 0&1&2\\\hline 1&1&3\\\hline 1\\\cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 0&0&1\\\hline 0&0&3\\\hline 0\\\cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 0&0&1\\\hline 0&0&2\\\hline 0\\\cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 0&0&1\\\hline 0&0&1\\\hline 0\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&0&0\\\hline 0\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&0&0\\\hline 1\\\cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1&0&0\\\hline 0&0&0\\\hline 1\\\cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1&0&0\\\hline 0&0&1\\\hline 1\\\cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1&0&0\\\hline 0&0&2\\\hline 1\\\cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1&0&1\\\hline 0&0&2\\\hline 1\\\cline{1-1}\end{array} =: Array \quad A = \Phi(P)$$

$Weight(P) = 0 + 1 + 2 + 1 + 1 + 3 + 2 = 10 = \sum_{i,j} A_{i,j} hook(i,j) =$
$1 * 5 + 1 * 2 + 2 * 1 + 1 * 1 = weight(A)$
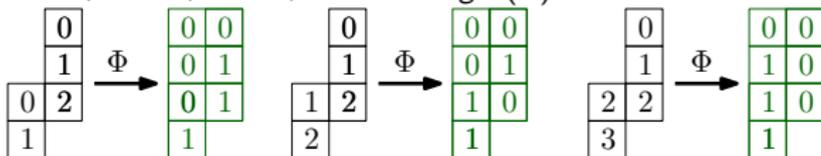
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$1*5 + 1*2 + 2*1 + 1*1 = weight(A)$

Theorem (Morales-Pak-P)

The restricted Hillman-Grassl map is a bijection to the SSYTs of shape λ/μ to the excited arrays (diagrams in $\mathcal{E}(\lambda/\mu)$ with nonzero entries on the broken diagonals) .
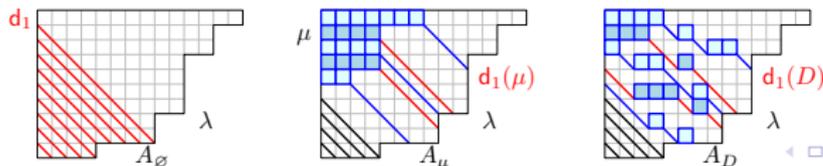
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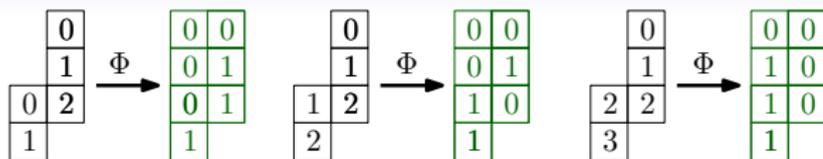
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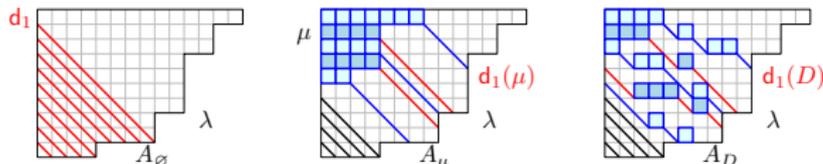
Proof sketch:

Issue: enforce 0s on μ and strict increase down columns on λ/μ.

Show $\Phi^{-1}(A)$ is column strict in λ/μ + support in λ/μ via properties of RSK (each diagonal of P is shape of RSK tableau on the corresponding subrectangle of A)

Thus, Φ^{-1} is injective: restricted arrays $\rightarrow$ SSYTs of shape λ/μ.

Bijective: use the algebraic identity.

Hillman-Grassl on skew RPPs

Without the restriction of strictly increasing columns, we have skew reverse plane partitions and a wider class of arrays/diagrams, called *pleasant diagrams*: $PD(\lambda/\mu)$.
– supersets of $\mathcal{E}(\lambda/\mu)$, identified by the "high peaks".

λ/μ S $D^* = \varrho_1(S)$ $D = \varrho_2(S)$
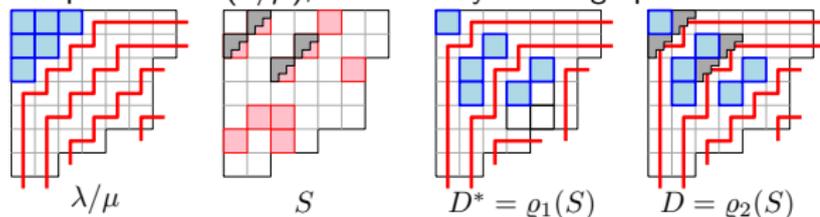
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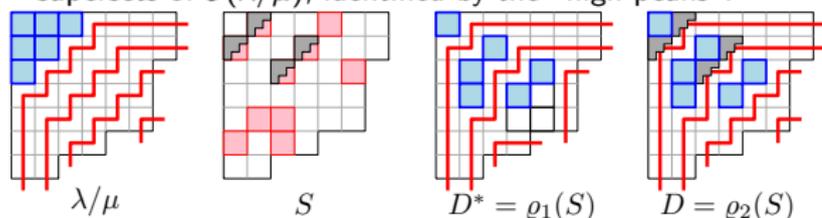
Theorem (MPP)

The HG map is a bijection between skew RPPs of shape λ/μ and arrays with certain nonzero entries (at the "high peaks"):

$$\sum_{\pi \in RPP(\lambda/\mu)} q^{|\pi|} = \sum_{S \in PD(\lambda/\mu)} \prod_{u \in S} \left[\frac{q^{h(u)}}{1 - q^{h(u)}} \right].$$
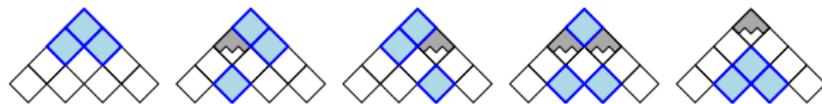
With P-partitions/limit: combinatorial proof of original Naruse Hook-Length Formula for $f^{\lambda/\mu}$..

2^7 2^6 2^6 2^5 2^6

Non-intersecting lattice paths

Theorem[Lascoux-Pragacz, Hamel-Goulden] If $(\theta_1, \ldots, \theta_k)$ is a Lascoux–Pragacz decomposition (i.e. maximal outer border strip decomposition) of λ/μ, then

$$s_{\lambda/\mu} = \det \left[s_{\theta_i \# \theta_j} \right]_{i,j=1}^k.$$

where $s_\varnothing = 1$ and $s_{\theta_i \# \theta_j} = 0$ if the $\theta_i \# \theta_j$ is undefined.

(Here θ_1 is the border strip following the inner border of λ and θ_i are obtained from the inner border of the remaining partition, until μ is hit, then the border strips are obtained from each connected part etc, and ordered by their corners. The strip $\theta_i \# \theta_j$ is the shape of θ_1 between the diagonals of the endpoints of θ_i and θ_j.)
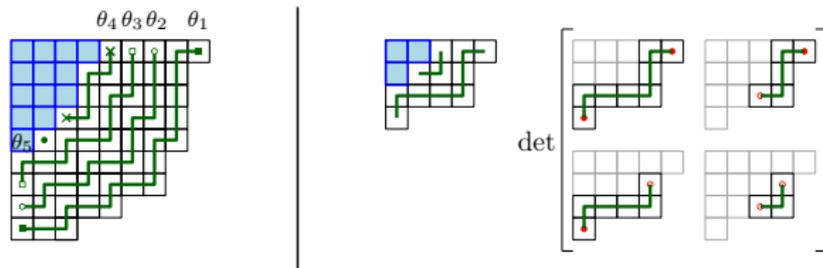
NHLF for border strips

Lemma (MPP)

For a border strip $\theta = \lambda/\mu$ with end points (a,b) and (c,d) we have

$$s_\theta(1, q, q^2, \ldots) = \sum_{\substack{\gamma:(a,b)\to(c,d) \\ \gamma \subseteq \lambda}} \prod_{(i,j)\in\gamma} \frac{q^{\lambda'_j - i}}{1 - q^{h(i,j)}}.$$

$s_{\scriptsize\square}(1, q, q^2, \ldots) = \underset{\gamma=(3,1),(3,2),(2,2),(2,3),(1,3):}{\frac{q^3}{(1-q^2)(1-q^1)(1-q^3)(1-q^1)(1-q^2)}} + \frac{q^4}{(1-q)(1-q^2)^2(1-q^3)(1-q^4)}$

$+ \frac{q^1}{(1-q)(1-q^2)^2(1-q^3)(1-q^4)} + \frac{q^7}{(1-q)^2(1-q^3)(1-q^4)^2} + \frac{q^6}{(1-q)^2(1-q^5)(1-q^4)^2}$

Proofs: induction on $|\lambda/\mu|$, or [multivariate] Chevalley formula for factorial Schurs.

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Excited diagrams for λ/μ – NonIntersecting Lattice Paths:
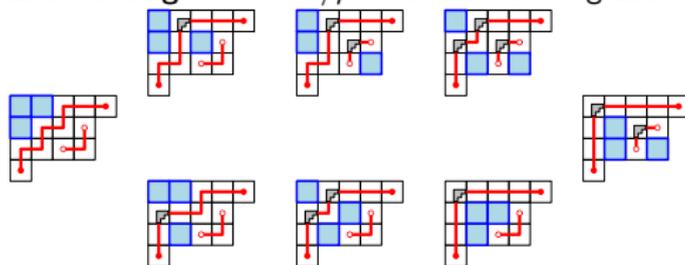
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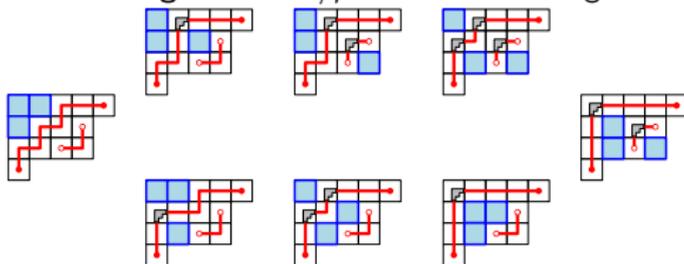
$$s_{\lambda/\mu} \overset{=}{\underset{\text{Lascoux-Pragacz}}{}} \det \left[s_{\theta_i \# \theta_j} \right]_{i,j=1}^{k} \overset{=}{\underset{s_{\text{Border Strip}}}{}} \det \left[\sum_{\gamma:(a_i,b_i)\to(c_j,d_j)} \prod_{u\in\gamma} \frac{q^{\cdot\cdot}}{1 - q^{h_u}} \right]$$

$$\overset{=}{\underset{Lindstrom-Gessel-Viennot}{}} \sum_{NILP:\gamma_1,\dots} \prod_{u\in\gamma_1\cup\dots} \frac{q^{\cdot\cdot}}{1 - q^{h_u}} \overset{\mathcal{E}(\lambda/\mu)=NILP}{=} \sum_{D\in ED(\lambda/\mu)} \prod_{u\in D} \frac{q^{\cdot\cdot\cdot}}{1 - q^{h_u}}$$

Asymptotics of the number of skew SYTs

$\lambda/\mu =$, $\quad |\lambda/\mu| = n \to \infty$

 $f^{\lambda/\mu} = n!$ $\qquad f^{\lambda/\emptyset} = C_{n/2} \approx 2^n \frac{2\sqrt{2}}{n^{3/2}\sqrt{\pi}}$

Question: What is the asymptotic value of $f^{\lambda/\mu}$, $|\lambda/\mu| = n$ as $n \to \infty$ and λ, μ change under various regimes:

0. If $\mu = \emptyset$, then $f^\lambda \sim \sqrt{n!}(1 + O(1/n))$ for $\lambda \sim$ Plancherel.

1. [Stanley, 2001]: when μ is fixed, $\lambda^n \to (a; b)$ (Frobenius limit):

$$f^{\lambda^n/\mu} \sim f^{\lambda^n} s_\mu(\rho_a^+; \rho_b^-)(1 + O(1/n)),$$

where ρ_a^+, ρ_b^- are the corresponding specializations.
Similar results in [Corteel-Goupil-Schaeffer]
[Okounkov-Olshanski]

10

Tool

Naruse Hook-Length formula:

$$f^{\lambda/\mu} = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in D} \frac{1}{h_u}.$$

Define the "naive" hook-length formula:

$$F(\lambda/\mu) := \prod_{u \in \lambda/\mu} \frac{1}{h_u}.$$
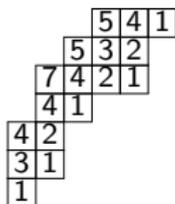
 $\quad F((6,5,5,3,2,2,1)/(3,2,1,1)) = \frac{1}{5\cdot4\cdot1\cdot5\cdot3\cdot2\cdot7\cdot4\cdot2\cdot1\cdot4\cdot1\cdot4\cdot2\cdot3\cdot1\cdot1}$

Corollary

$$F(\lambda/\mu) \leq f^{\lambda/\mu} \leq |\mathcal{E}(\lambda/\mu)| F(\lambda/\mu)$$

General bounds: size of $\mathcal{E}(\lambda/\mu)$

$$F(\lambda/\mu) \leq f^{\lambda/\mu} \leq |\mathcal{E}(\lambda/\mu)| F(\lambda/\mu)$$

$\mathcal{E}(\lambda/\mu) = \{$ Non-intersecting Lattice Paths in λ/μ $\}$
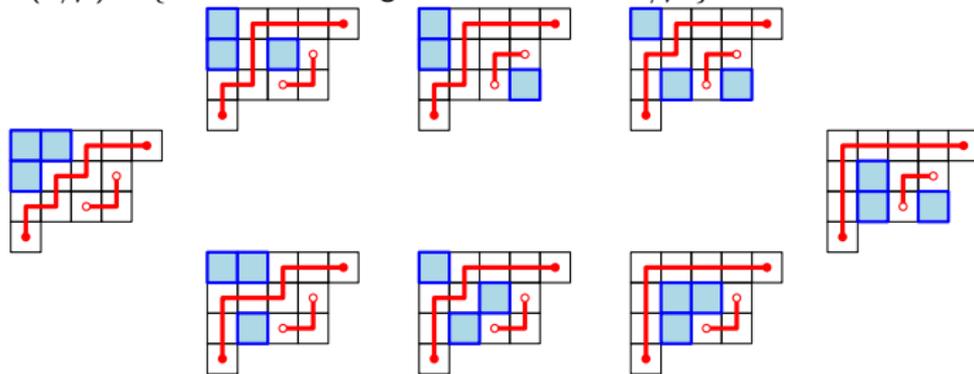
Lemma (MPP)

If $|\lambda/\mu| = n$ then $\mathcal{E}(\lambda/\mu) \leq 2^n$.

Lemma (MPP)

If d is the Durfee square size of λ, then $\mathcal{E}(\lambda/\mu) \leq n^{2d^2}$.

The "linear" regime
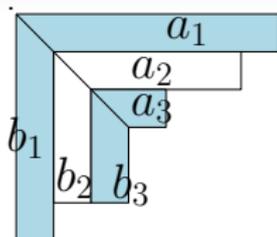
$a(\lambda) = (a_1, a_2, \ldots)$, $b(\lambda) = (b_1, b_2, \ldots)$ – Frobenius coordinates of λ. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\beta := (\beta_1, \ldots, \beta_k)$ be fixed sequences in $\mathbb{R}_+^k$.

Thoma–Vershik–Kerov (TVK) *limit* if $a_i/n \to \alpha_i$ and $b_i/n \to \beta_i$ as $n \to \infty$, for all $1 \le i \le k$.

Theorem (MPP)

Let $\{\lambda^{(n)}/\mu^{(n)}\}$ be a sequence of skew shapes with a TVK limit, i.e. suppose $\lambda^{(n)} \to (\alpha, \beta)$, where $\alpha_1, \beta_1 > 0$, and $\mu^{(n)} \to (\pi, \tau)$ for some $\alpha, \beta, \pi, \tau \in \mathbb{R}_+^k$. Then

$$\log f^{\lambda^{(n)}/\mu^{(n)}} = cn + o(n) \quad as \quad n \to \infty,$$

where

$$c = \gamma \log \gamma - \sum_{i=1}^{k}(\alpha_i - \pi_i) \log(\alpha_i - \pi_i) - \sum_{i=1}^{k}(\beta_i - \tau_i) \log(\beta_i - \tau_i)$$

and

$$\gamma = \sum_{i=1}^{k}(\alpha_i + \beta_i - \pi_i - \tau_i).$$

The stable shape: $\sqrt{n}$ scale
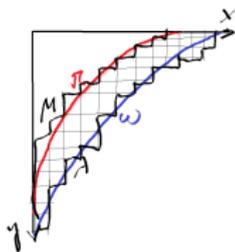
Theorem (MPP)

Let $\omega, \pi : [0, a] \to [0, b]$ be continuous non-increasing functions, and suppose that area$(\omega/\pi) = 1$. Let $\{\lambda^{(n)}/\mu^{(n)}\}$ be a sequence of skew shapes with the stable shape ω/π, i.e. $[\lambda^{(n)}]/\sqrt{n} \to \omega$, $[\mu^{(n)}]/\sqrt{n} \to \pi$. Then

$$\log f^{\lambda^{(n)}/\mu^{(n)}} \sim \frac{1}{2} n \log n \quad as \quad n \to \infty.$$
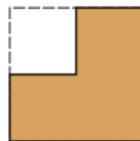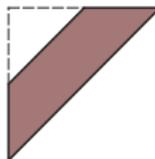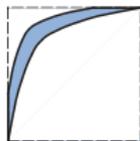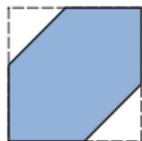
The stable shape: $\sqrt{n}$ scale
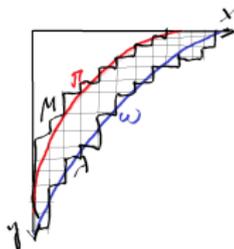
Theorem (MPP)

Suppose $(\sqrt{N} - L)\omega \subset [\lambda^{(n)}](\sqrt{N} + L)\omega$ for some $L > 0$, and similarly for $\mu^{(n)}$ wrt π, then

$$-(1 + c(\omega/\pi))n + o(n) \leq \log f^{\lambda^{(n)}/\mu^{(n)}} - \frac{1}{2} n \log n \leq -(1 + c(\omega/\pi))n + \log \mathcal{E}(\lambda^{(n)}/\mu^{(n)}) + o(n),$$

as $n \to \infty$, where

$$c(\omega/\pi) = \iint_{\omega/\pi} \log h(x, y) dx dy,$$

where $h(x, y)$ is the hook length from (x, y) to ω.

14

Subpolynomial depth, "thin" shapes
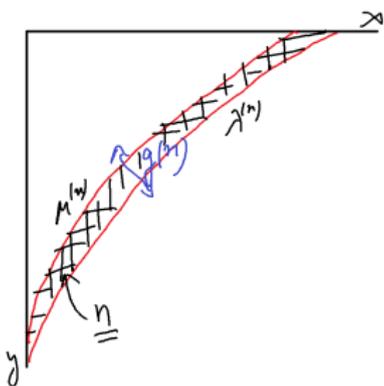
Suppose
$depth := \max_{u \in \lambda/\mu} h_u =: g(n) = n^{o(1)}$
(subpolynomial growth).

Theorem (MPP)

Let $\{\nu_n = \lambda^{(n)}/\mu^{(n)}\}$ be a sequence of skew partitions with a subpolynomial depth shape associated with the function $g(n)$. Then

$$\log f^{\nu_n} = n \log n - \Theta\big(n \log g(n)\big) \quad \text{as} \quad n \to \infty.$$

Thick ribbons
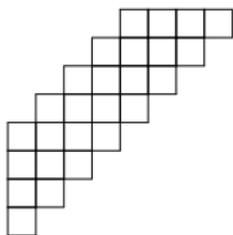
Theorem (MPP)

Let $\gamma_k := (\delta_{2k}/\delta_k)$, where $\delta_k = (k-1, k-2, \ldots, 2, 1)$. Then

$$\frac{1}{6} - \frac{3}{2}\log 2 + \frac{1}{2}\log 3 + o(1) \leq \frac{1}{n}\left(\log f^{\gamma_k} - \frac{1}{2}n\log n\right) \leq \frac{1}{6} - \frac{7}{2}\log 2 + 2\log 3 + o(1),$$

where $n = |\gamma_k| = k(3k-1)/2$.

Update: There exists a c, s.t. $c = \lim_{n\to\infty} \frac{1}{n}\left(\log f^{\gamma_k} - \frac{1}{2}n\log n\right)$.
Jay Pantone's implementation (method of differential approximants) on 150+ terms
of the sequence $\{\log f^{\gamma_k}\}$ to approximate $c \approx -0.1842$.

Thin ribbons
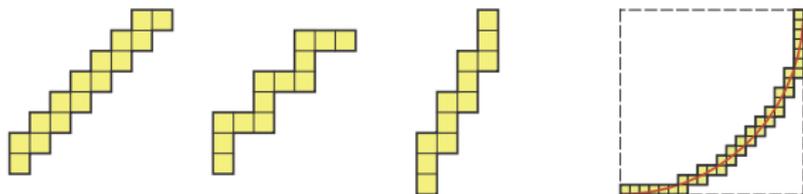
Zigzag: $\rho_k := \delta_{k+2}/\delta_k$, $E_n = |\{\sigma \in S_n : \sigma(1) < \sigma(2) > \sigma(3) < \cdots\}|$ – Euler numbers, alternating permutations.

$$f^{\rho_n} = E_{2n+1}; \qquad E_m \sim m!(2/\pi)^m 4/\pi(1 + o(1))$$

From theorem: $F(\rho_k) = n!/3^k$, $\mathcal{E}(\rho_k) = C_k$, so

$$\frac{(2k+1)!}{3^k} \leq E_{2k+1} \leq \frac{(2k+1)!C_k}{3^k}$$

Problem: If $\gamma_n := \lambda/\mu$ is a border strip (ribbon of thickness 1, n boxes) approaching a given curve γ under rescaling by n, what is $\log f^{\gamma_n} - n \log n$ in terms of γ? Is it true that $\frac{\log f^{\gamma_n} - n \log n}{n} \to c(\gamma)$ for some constant $c(\gamma)$? (Permutations with certain descent sequences)

17
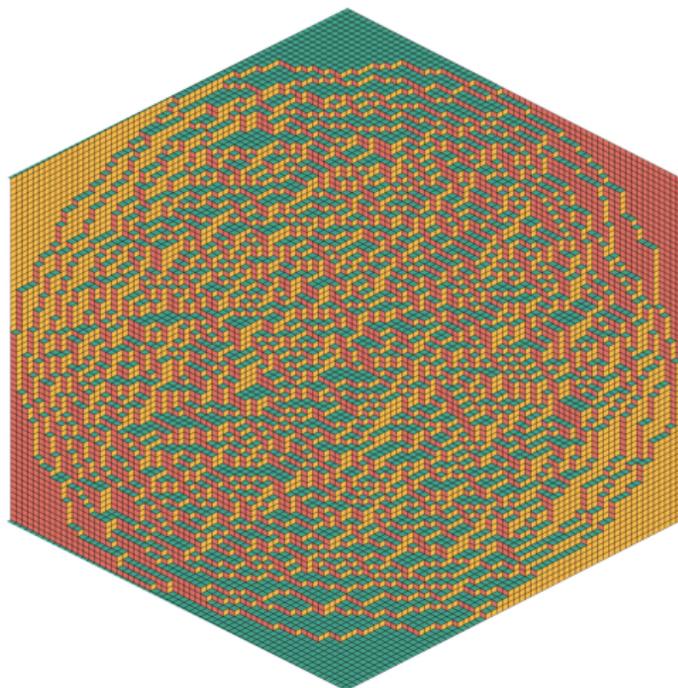
Lozenge tilings

Image: Leonid Petrov

Lozenge tilings with multivariate weights

Plane partitions with base μ, height d

weights of horizontal lozenges $= x_i - y_j$
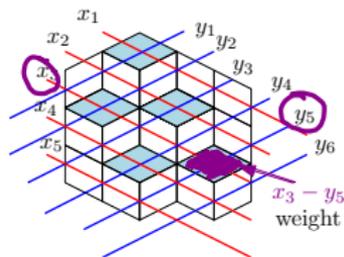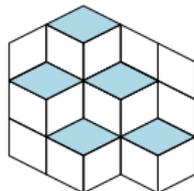
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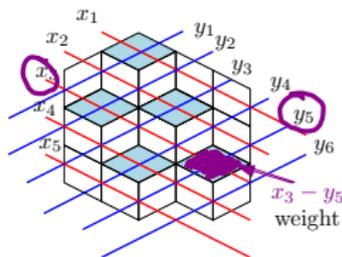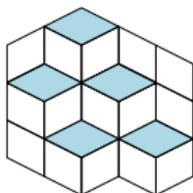
Theorem (Morales-Pak-P)

Consider tilings with base μ and height d, we have that

$$\sum_{T \in \Omega_{\mu,d}} \prod_{(i,j) \in T} (x_i - y_j) = \det[A_{i,j}(\mu, d)]_{i,j=1}^{d+\ell(\mu)},$$

where

$$A_{i,j}(\mu, d) := \begin{cases} \frac{(x_i - y_1) \cdots (x_i - y_{d+\ell(\mu)-j})}{(x_i - x_{i+1}) \cdots (x_i - x_{d+\ell(\mu)})}, & \text{when } j = \ell(\mu) + 1, \ldots, \ell(\mu) + d, \\ \frac{(x_i - y_1) \cdots (x_i - y_{\mu_j + d})}{(x_i - x_{i+1}) \cdots (x_i - x_{d+j})}, & \text{when } j = i - d, \ldots, \ell(\mu), \\ 0, & \text{when } j < i - d. \end{cases}$$

Theorem (Morales-Pak-P)

Consider tilings of the $a \times b \times c \times a \times b \times c$ (base $a \times b$, height c) hexagon with horizontal lozenges having weights $x_i - y_j$, i.e. tilings $\Omega_{a,b,c}$ with rectangular base $\mu = a \times b$ and height c. The partition function is given by

$$Z(a,b,c) := \sum_{T \in \Omega_{a,b,c}} \prod_{(i,j) \in T} (x_i - y_j) = \det \left[\begin{cases} \frac{(x_i - y_1) \cdots (x_i - y_{c+a-j})}{(x_i - x_{i+1}) \cdots (x_i - x_{c+a})} & \text{if } j > a \\ \frac{(x_i - y_1) \cdots (x_i - y_{b+c})}{(x_i - x_{i+1}) \cdots (x_i - x_{c+j})} & \text{if } j = i - c, \dots, a \\ 0, & j < i - c \end{cases} \right]_{i,j=1}^{a+c}$$

Consider a path $P(d_1, \dots)$ consisting of vertical lozenges (i.e. not the horizontal lozenges) passing through the points (i, d_i) (ith vertical line, distance of the midpoint $d_i + 1/2$ from the top axes) (necessarily $|d_i - d_{i+1}| \leq 1$, $d_i \leq d_{i+1}$ if $i \leq b$ and $d_i \geq d_{i+1}$ if $i > b$, and $d_1 = d_{a+b}$).
The probability that such path exists is given by

$$\frac{\det[A_{i,j}(\mu, d)] \det[\bar{A}_{i,j}(\bar{\mu}, c - d - 1)]}{Z}$$

where $d := d_1$, $\ell(\mu) = b$, $\mu_1 = a$ and μ is given by its diagonals $-(d_1 - d, d_2 - d, \dots)$, and $\bar{\mu}$ is the complement of μ in $a \times b$. The matrix $\bar{A}$ is defined as in previous Theorem with the substitution of x_i by $x_{a+c+1-i}$ and y_j by $y_{b+c+1-j}$.
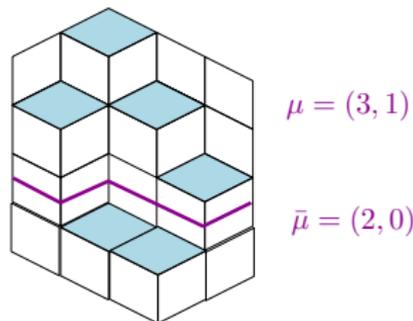
$\mu = (3, 1)$

$\bar{\mu} = (2, 0)$

Excited diagrams and factorial Schur functions

Factorial Schur functions.

$$s_\mu^{(d)}(x|a) := \frac{\det\left[(x_j - a_1)\cdots(x_j - a_{\mu_i+d-i})\right]_{i,j=1}^d}{\prod_{1 \le i < j \le d}(x_i - x_j)},$$

where $x = (x_1, x_2, \ldots, x_d)$ and $a = (a_1, a_2, \ldots)$ is a sequence of parameters.

Excited diagrams $\mathcal{E}(\lambda/\mu)$: Start with λ/μ. Move cells of μ inside λ via:
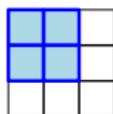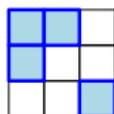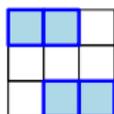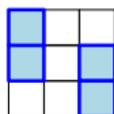
$q^3 \qquad q^4 \qquad q^5 \qquad q^5 \qquad q^6 \qquad q^7$

Excited diagrams and factorial Schur functions

Factorial Schur functions.

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q^3 q^4 q^5 q^5 q^6 q^7
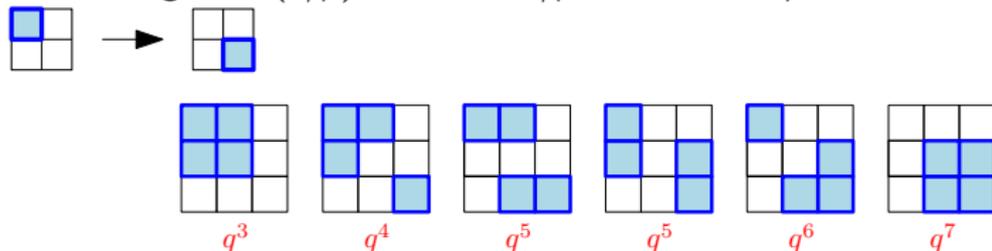
Theorem (Ikeda-Naruse Multivariate "Hook-Length Formula")

Let $\mu \subset \lambda \subset d \times (n - d)$. Let v be the Grassmannian permutation with unique descent at position d corresponding to λ, i.e. $v(d' + 1 - i) = \lambda_i + (d' + 1 - i)$ and $v(j) = d' + j - \lambda_j'$. Then

$$s_\mu^{(d)}(y_{v(1)}, \ldots, y_{v(d)}|y_1, \ldots, y_{n-1}) = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in D} (y_{v(d-i+1)} - y_{v(d+j)})$$

Excited diagrams and factorial Schur functions

Excited diagrams $\mathcal{E}(\lambda/\mu)$: Start with λ/μ. Move cells of μ inside λ via:
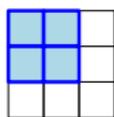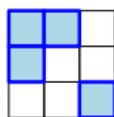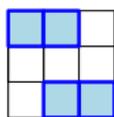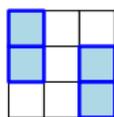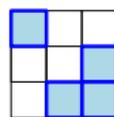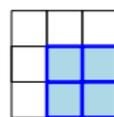
$$q^3 \qquad q^4 \qquad q^5 \qquad q^5 \qquad q^6 \qquad q^7$$

Theorem (Ikeda-Naruse Multivariate "Hook-Length Formula")

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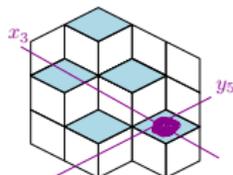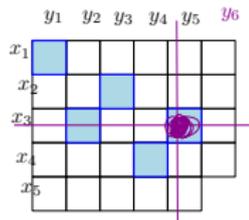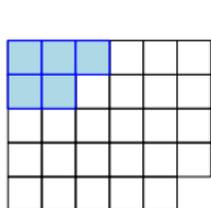
Product formulas
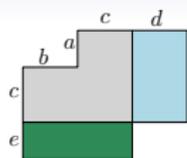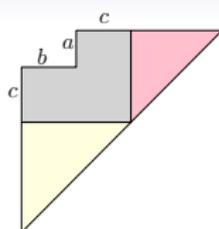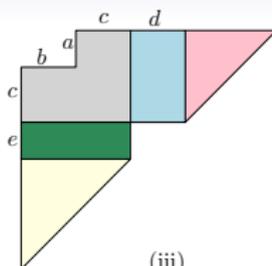
(i)   (ii)   (iii)

$\Phi(n) := 1! \cdot 2! \cdots (n-1)!$, $\Psi(n) := 1!! \cdot 3!! \cdots (2n-3)!!$,
$\Psi(n; k) := (k+1)!! \cdot (k+3)!! \cdots (k+2n-3)!!$, $\Lambda(n) := (n-2)!(n-4)! \cdots$

Theorem (MPP)

For nonnegative integers a, b, c, d, e, let n be the size of the corresponding skew shape, then for the shapes in (i), (ii), (iii) we have the following product formulas for the number of skew SYTs:

$$f^{sh(i)} = n! \ \frac{\Phi(a)\Phi(b)\Phi(c)\Phi(d)\Phi(e)\Phi(a+b+c)\Phi(c+d+e)\Phi(a+b+c+e+d)}{\Phi(a+b)\Phi(e+d)\Phi(a+c+d)\Phi(b+c+e)\Phi(a+b+2c+e+d)},$$

$$f^{sh(ii)} = n! \ \frac{\Phi(a)\Phi(b)\Phi(c)\Phi(a+b+c)}{\Phi(a+b)\Phi(b+c)\Phi(a+c)} \ \frac{\Psi(c)\Psi(a+b+c)}{\Psi(a+c)\Psi(b+c)\Psi(a+b+2c)},$$

$$f^{Sh(iii)} = \frac{n! \quad \Phi(a)\Phi(b)\Phi(c)\Phi(a+b+c)\ \Psi(c; d+e)\Psi(a+b+c; d+e)\ \Lambda(2a+2c)\Lambda(2b+2c)}{\Phi(a+b)\Phi(b+c)\Phi(a+c)\Psi(a+c)\Psi(b+c)\Psi(a+b+2c; d+e)\Lambda(2a+2c+d)\Lambda(2b+2c+e)}$$

Merci

Merci

Happy Birthday Jean-Yves Thibon!