# Insertion algorithms for shifted domino tableaux

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Séminaire Lotharingien de Combinatoire

# Plan

1. Shifted domino tableaux
2. Insertion algorithms

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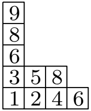
# Introduction

Young tableaux: (Young)
- Schur functions
- Plactic monoid (Lascoux, Schützenberger)

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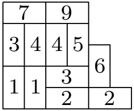
Domino tableaux: (Young)
- Product of two Schur functions
- Super plactic monoid (Carré, Leclerc)

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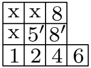
Shifted Young tableaux: (Sagan, Worley)
- P- and Q-Schur functions
- Shifted plactic monoid (Serrano)

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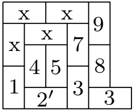
Shifted domino tableaux : (Chemli)
- Product of two P- and Q-Schur function
- Super shifted plactic monoid



# Young tableaux

A partition $\lambda$ of $n$ is a non-increasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. We represent a partition by its Ferrers diagram.

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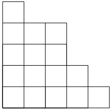

Figure: The Ferrers diagram of $\lambda$=(5,4,3,3,1)

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| | | | | |
|---|---|---|---|---|
| 9 | | | | |
| 5 | 7 | 9 | | |
| 4 | 5 | 5 | | |
| 2 | 3 | 4 | 6 | |
| 1 | 1 | 3 | 4 | 7 |

Figure: A Young tableau of shape $\lambda$=(5,4,3,3,1)

A Young tableau is a filling of a Ferrers diagram with positive integers such that rows are non-decreasing and columns are strictly increasing.

# Domino tilling

Two adjacent boxes form a domino:

 or 

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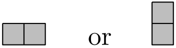
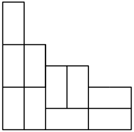
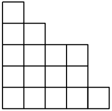


A diagram is tileable if we can tile it by non intersecting dominos.



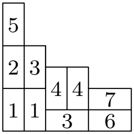
tileable



non tileable

# Domino tableaux

Given a tiled partition $\lambda$, a domino tableau is a filling of dominos with positive integers such that columns are strictly increasing and rows are non decreasing.



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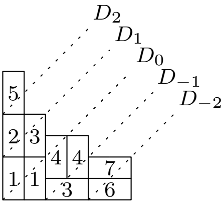


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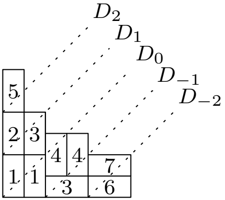


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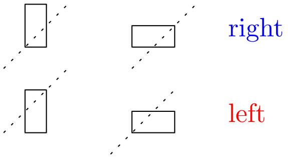
2 types of dominos:



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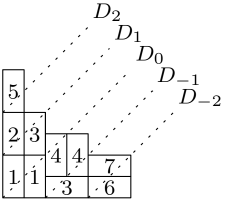


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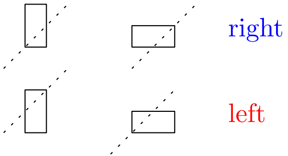


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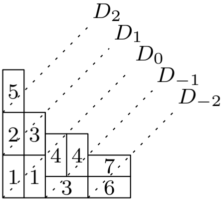


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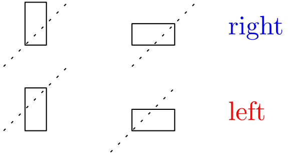
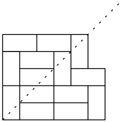
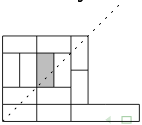
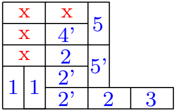



We do not allow tillings such that we can remove a domino strictly above $D_0$ and obtain a domino tableau.
A tilling is acceptable iff there is no vertical domino $d$ on $D_0$ such that the only domino adjacent to $d$ on the left is strictly above $D_0$.

acceptable 

 not acceptable

# Shifted domino tableaux




Given an acceptable tilling, a shifted domino tableau is:

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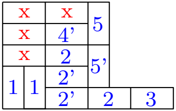


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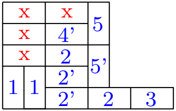


Given an acceptable tilling, a shifted domino tableau is:

- a filling of dominos strictly above $D_0$ by x
- a filling of other dominos with integers in $\{1' < 1 < 2' < 2 < \cdots\}$
- columns and rows are non decreasing
- an integer without ' appears at most once in every column
- an integer with ' appears at most once in every row

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# Insertion algorithm

We consider bicolored words of positive integers, namely elements of $(\mathbb{N}^* \times \{L, R\})^*$, for exemple w= 123232

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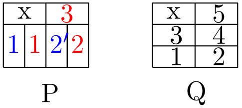
**Theorem (Chemli, P. (2016))**

*There is a bijective algorithm $f$, with a bicolored word as input and a pair $(P, Q)$ of shifted domino tableaux as output such that:*

- *$P$ and $Q$ have same shape*

$w = 13212$



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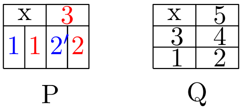

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- *$P$ and $Q$ have same shape*
- *$P$ is without ′ on $D_0$*
- *$Q$ is standard without ′*

$w = 13212$

| | | | |
|---|---|---|---|
| x | | 3 | |
| 1 | 1 | 2′ | 2 |

P

| | |
|---|---|
| x | 5 |
| 3 | 4 |
| 1 | 2 |

Q

# Algebraic consequences

## Theorem (Chemli, P. (2016))

*Let $w_1$ be a word in $\mathbb{N} \times \{L\}$ with P-tableau of shape $\mu$ , and $w_2$ be a word in $\mathbb{N} \times \{R\}$ with P-tableau of shape $\nu$. Let $\lambda$ be the shape of the P-tableau of the word $w_1 w_2$. We have:*

$$\sum_{T,\ sh(T)=\lambda} x^T = P_\mu P_\nu$$

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**Theorem (Chemli, P. (2016))**

*Two words belong to the same class of the super shifted plactic monoid iff they have the same P-tableau.*

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*There is an algorithm $g$ with a bicolored standard word as input and a pair $(P, Q)$ of shifted domino tableaux as output*

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- *$P$ is standard without '*
- *$Q$ is standard without ' in $D_0$*

# Conjectures

## Conjecture 1

If $\sigma$ is a signed permutation (that we identify with a bicolored standart word) then

$$f(\sigma) = (P, Q) \Leftrightarrow g(\sigma^{-1}) = (Q, P)$$

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## Conjecture 2

Algorithm $f$ commutes with standardization and truncation.

# What is missing?

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- Cauchy identity
- Hook formula for shifted domino tableaux

Thank you for your attention!