Difference operators for functions of partitions and its application to hook-content identities

(joint with Paul-Olivier Dehaye and Guo-Niu Han)

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Definitions

- partition: $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$.
- size: $\mid \lambda \mid = \sum_{1 \leq i \leq \ell} \lambda_i$.
- Young diagram: boxes arranged in left-justified rows with λ_i boxes in the i-th row.
- hook length: $h_\square :=$ # boxes exactly to the right, exactly above, and $\square$ itself.
- $H(\lambda)$: the product of all hook lengths in the Young diagram.

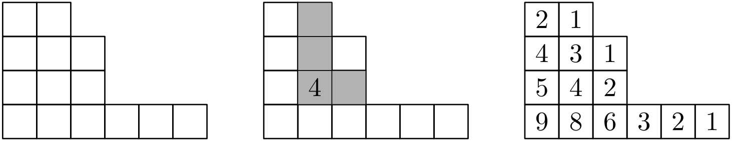

Figure: The Young diagram of the partition $(6, 3, 3, 2)$ and the hook lengths of corresponding boxes.

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- content: $c_\square := j - i$ for the box $\square$ in the i-th row and j-th column.

-3	-2				
-2	-1	0			
-1	0	1			
0	1	2	3	4	5

Figure: The contents of the partition $(6, 3, 3, 2)$.

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- standard Young tableau (SYT) of the shape λ: fill in the Young diagram with distinct numbers 1 to $|\lambda|$ such that the numbers in each row and each column are increasing.
- f_λ: # SYTs of the shape λ.

6	9				
3	8	14			
2	5	13			
1	4	7	10	11	12

Figure: A standard Young tableau of the shape $(6, 3, 3, 2)$.

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$$\sum_{n\geq 0}\frac{x^n}{n!^2}\left(\sum_{|\lambda|=n} f_\lambda^2 \prod_{\square\in\lambda}(y+h_\square^2)\right) = \prod_{i\geq 1}(1-x^i)^{-1-y}.$$

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Theorem (Han 2008)

Let $\mathcal{H}_t(\lambda)$ be the multiset of the hook lengths of λ which are divisible by t. Then

$$\sum_{\lambda\in\mathcal{P}} x^{|\lambda|}\prod_{h\in\mathcal{H}_t(\lambda)}\left(y-\frac{tyz}{h^2}\right) = \prod_{k\geq 1}\frac{(1-x^{tk})^t}{(1-(yx^t)^k)^{t-z}(1-x^k)}.$$

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- The case $z=0,\ y=1$ gives the generating function for the number of partitions.
- Another corollary is the Marked hook formula:

$$\frac{1}{n!}\sum_{|\lambda|=n} f_\lambda^2 \sum_{h\in\mathcal{H}(\lambda)} h^2 = \frac{n(3n-1)}{2}.$$

- $\frac{f_\lambda^2}{|\lambda|!}$ is called the Plancherel measure of the partition λ.
- $\frac{1}{n!}\sum_{|\lambda|=n} f_\lambda^2 g(\lambda)$ is called the n-th Plancherel average of the function $g(\lambda)$.
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For which function $g(\lambda)$, its Plancherel average $\frac{1}{n!}\sum_{|\lambda|=n} f_\lambda^2 g(\lambda)$ has a nice expression?

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Han 2008

- $\frac{1}{n!}\sum_{|\lambda|=n} f_\lambda^2 \sum_{\square\in\lambda} h_\square^2 = \frac{3n^2-n}{2}$.
- $\frac{1}{n!}\sum_{|\lambda|=n} f_\lambda^2 \sum_{\square\in\lambda} h_\square^4 = \frac{40n^3-75n^2+41n}{6}$.
- $\frac{1}{n!}\sum_{|\lambda|=n} f_\lambda^2 \sum_{\square\in\lambda} h_\square^6 = \frac{1050n^4-4060n^3+5586n^2-2552n}{24}$.

Conjecture (Han 2008)

The Plancherel average of the function $g(\lambda) = \sum_{\square \in \lambda} h_\square^{2k}$:

$$P(n) = \frac{1}{n!} \sum_{|\lambda|=n} f_\lambda^2 \sum_{\square \in \lambda} h_\square^{2k}$$

is always a polynomial of n for every $k \in \mathbb{N}$.

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- This conjecture was proved and generalized by Stanley.

Theorem (Stanley 2010)

Let Q_1 and Q_2 be two given symmetric functions. Then the Plancherel average of the function $Q_1(h_\square^2 : \square \in \lambda)Q_2(c_\square : \square \in \lambda)$:

$$P(n) = \frac{1}{n!} \sum_{|\lambda|=n} f_\lambda^2 Q_1(h_\square^2 : \square \in \lambda)Q_2(c_\square : \square \in \lambda)$$

is a polynomial of n.

- Olshanski (2010) also proved the content case.

- An application of Han-Stanley Theorem:

Corollary (Okada-Panova 2008)

$$n! \sum_{|\lambda|=n} \frac{\sum_{\square\in\lambda} \prod_{i=1}^{r} (h_{\square}^2 - i^2)}{H(\lambda)^2} = \frac{1}{2(r+1)^2} \binom{2r}{r} \binom{2r+2}{r+1} \prod_{j=0}^{r} (n-j).$$

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Definition

Let $g(\lambda)$ be a function defined on partitions. The difference operator D on functions of partitions is defined by

$$Dg(\lambda) := \sum_{|\lambda^+/\lambda|=1} g(\lambda^+) - g(\lambda).$$

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- The coefficient on the right hand side of Okada-Panova formula can be obtained by letting the difference operator act on one single partition:

$$H_\lambda D^{r+1}\left(\frac{\sum_{\square\in\lambda}\prod_{1\le j\le r}(h_\square^2-j^2)}{H_\lambda}\right)=\frac{1}{2(r+1)^2}\binom{2r}{r}\binom{2r+2}{r+1}.$$

Figure: The poset of nonnegative integers.

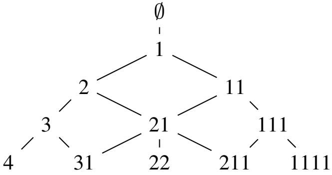

Figure: Young's lattice (the poset of partitions).

- $\Delta g(x) := g(x+1) - g(x)$.
- $\Delta^r g(x) = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} g(x+i)$.
- $g(x)$ is a polynomial iff $\Delta^{r+1} g(x) = 0$ for some r.
- Basis of polynomials: $\{g(x) = x^k : k \in \mathbb{N}\}$.
- Other posets: posets of (1) partitions, (2) partitions with the given t-core, (3) self-conjugate partitions, (4) doubled distinct partitions, (5) strict partitions?

- $Dg(\lambda) := \sum_{|\lambda^+/\lambda|=1} g(\lambda^+) - g(\lambda)$.
- $D^n g(\mu) = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \sum_{|\lambda/\mu|=k} f_{\lambda/\mu} g(\lambda)$.
- $\sum_{|\lambda/\mu|=n} f_{\lambda/\mu} g(\lambda) = \sum_{k=0}^{n} \binom{n}{k} D^k g(\mu)$.
- $g(\lambda)$ is a D-polynomial iff $D^{n+1} g(\lambda) = 0$ for some n.
- Basis of D-polynomials? hard to characterize!
- We show that $\frac{Q_1(h_\square^2 : \square \in \lambda) Q_2(c_\square : \square \in \lambda)}{H_\lambda}$ is always a D-polynomial (A long and technique proof). Therefore

$$\frac{1}{(n+|\mu|)!} \sum_{|\lambda/\mu|=n} f_\lambda f_{\lambda/\mu} Q_1(h_\square^2 : \square \in \lambda) Q_2(c_\square : \square \in \lambda)$$

is a polynomial of n.

- A partition λ is called a t*-core* partition if it has no hook length t.

The t-difference operator for function of partitions

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$(18, 7, 6)$

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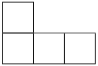

$(3, 1)$

- Let λ be a partition and g be a function defined on partitions. The t-difference operator D_t is defined by

$$D_t g(\lambda) := \sum_{\substack{\lambda^+ \geq_t \lambda \\ |\lambda^+/\lambda| = t}} g(\lambda^+) - g(\lambda).$$

- Example: $D_3 g((3,1)) = g((6,1)) + g((3,1,1,1,1)) + g((3,2,2)) - g((3,1))$.

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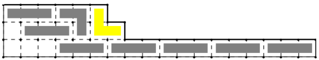

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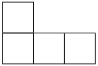
$$\overset{t=3}{\Longrightarrow}$$



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- Example: $D_3 g((3,1)) = g((6,1)) + g((3,1,1,1,1)) + g((3,2,2)) - g((3,1))$.
- $g(\lambda)$ is a D_t-polynomial iff $D_t^{r+1} g(\lambda) = 0$ for some r.
- Question: which functions are D_t-polynomials?

Main Theorem (X. 2015, joint with Dehaye and Han)

Suppose that t is a positive integer, $u', v', j_u, j'_v, k_u, k'_v$ are nonnegative integers and μ is a given partition. Then for every $r > \sum_{u=1}^{u'}(k_u+1) + \sum_{v=1}^{v'} \frac{k'_v+2}{2}$ we have

$$D_t^r\left(\frac{1}{H_t(\lambda)}\left(\prod_{u=1}^{u'}\sum_{\substack{\square\in\lambda\\ h_\square\equiv\pm j_u \,(\mathrm{mod}\, t)}} h_\square^{2k_u}\right)\left(\prod_{v=1}^{v'}\sum_{\substack{\square\in\lambda\\ c_\square\equiv j'_v \,(\mathrm{mod}\, t)}} c_\square^{k'_v}\right)\right) = 0$$

for every partition λ. Moreover,

$$P(n) := \sum_{\substack{\lambda\geq_t\mu\\ |\lambda/\mu|=nt}} \frac{F_{\lambda/\mu}}{H_t(\lambda)}\left(\prod_{u=1}^{u'}\sum_{\substack{\square\in\lambda\\ h_\square\equiv\pm j_u \,(\mathrm{mod}\, t)}} h_\square^{2k_u}\right)\left(\prod_{v=1}^{v'}\sum_{\substack{\square\in\lambda\\ c_\square\equiv j'_v \,(\mathrm{mod}\, t)}} c_\square^{k'_v}\right)$$

is a polynomial of n with degree at most $\sum_{u=1}^{u'}(k_u+1) + \sum_{v=1}^{v'} \frac{k'_v+2}{2}$.

Step 1 : We construct some complicated sets $A_k (k \geq 0)$ of functions of partitions such that $g \in A_{k+1}$ implies $D_t g \in A_k$. Finally $D_t^{k+1} g = 0$ if $g \in A_k$.

The outline of the proof of the main results

Step 1 : We construct some complicated sets $A_k (k \geq 0)$ of functions of partitions such that $g \in A_{k+1}$ implies $D_t g \in A_k$. Finally $D_t^{k+1} g = 0$ if $g \in A_k$.

Step 2 : Let k be a nonnegative integer and $0 \leq j \leq t-1$. Then

$$\frac{1}{H_t(\lambda)} \left(\prod_{u=1}^{u'} \sum_{\substack{\square \in \lambda \\ h_\square \equiv \pm j_u \pmod t}} h_\square^{2k_u} \right) \left(\prod_{v=1}^{v'} \sum_{\substack{\square \in \lambda \\ c_\square \equiv j'_v \pmod t}} c_\square^{k'_v} \right)$$

is in the set A_{r-1} for some r.

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Step 3 : By the above two steps we know there exists some $r \in \mathbb{N}$ such that

$$D_t^r \left(\frac{1}{H_t(\lambda)} \left(\prod_{u=1}^{u'} \sum_{\substack{\square \in \lambda \\ h_\square \equiv \pm j_u \pmod t}} h_\square^{2k_u} \right) \left(\prod_{v=1}^{v'} \sum_{\substack{\square \in \lambda \\ c_\square \equiv j'_v \pmod t}} c_\square^{k'_v} \right) \right) = 0$$

for every partition λ.

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Corollary

$$\frac{1}{(n+\mid \mu \mid)!} \sum_{|\lambda/\mu|=n} f_\lambda f_{\lambda/\mu} = \frac{1}{H(\mu)}.$$

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Corollary (Okada-Panova 2008)

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Corollary (Fujii-Kanno-Moriyama-Okada 2008)

$$n! \sum_{|\lambda|=n} \frac{\sum_{\square\in\lambda} \prod_{i=0}^{r-1} (c_\square^2 - i^2)}{H(\lambda)^2} = \frac{(2r)!}{(r+1)!^2} \prod_{j=0}^{r} (n-j).$$

- Corollaries of the main theorem for general t.

Corollary

Suppose that μ is a given t-core partition. Then we have

$$\sum_{\substack{\lambda_{t\text{-core}}=\mu \\ |\lambda/\mu|=nt}} \frac{F_{\lambda/\mu}}{H_t(\lambda)} \sum_{\substack{\square\in\lambda \\ h_\square\equiv 0 \pmod t}} h_\square^2 = nt^2 + 3t\binom{n}{2}.$$

Furthermore,

$$\sum_{\substack{\lambda_{t\text{-core}}=\mu \\ |\lambda/\mu|=nt}} \frac{F_{\lambda/\mu}}{H_t(\lambda)} \sum_{\square\in\lambda} h_\square^2 = \frac{3t^2n^2}{2} + \frac{nt(t^2-3t-1+24|\mu|)}{6} + \sum_{\square\in\mu} h_\square^2.$$

In particular, let $\mu=\emptyset$. We have

$$\sum_{\substack{\lambda_{t\text{-core}}=\emptyset \\ |\lambda|=nt}} \frac{n!\, t^n}{H_t(\lambda)^2} \sum_{\square\in\lambda} h_\square^2 = \frac{3t^2n^2}{2} + \frac{nt(t^2-3t-1)}{6}.$$

- Motivated by Han's proof of Nekrasov-Okounkov Formula, Pétréolle obtained the following results.

Theorem (Pétréolle 2015)

For any complex number z, the following formulas hold:

$$\left(\prod_{i\geq 1}\frac{(1-x^{2i})^{z+1}}{1-x^i}\right)^{2z-1}=\sum_{\lambda\in SC}\delta_\lambda\, x^{|\lambda|}\prod_{h\in\mathcal{H}(\lambda)}\left(1-\frac{2z}{h\,\varepsilon_h}\right),$$

$$\prod_{k\geq 1}(1-x^k)^{2z^2+z}=\sum_{\lambda\in DD}\delta_\lambda\, x^{|\lambda|/2}\prod_{h\in\mathcal{H}(\lambda)}\left(1-\frac{2z+2}{h\,\varepsilon_h}\right),$$

where the sum is over all self-conjugate and doubled distinct partitions respectively.

Self-conjugate partitions

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- self-conjugate partition: a partition whose Young diagram is symmetric along the main diagonal.
- $\mathcal{SC}$: the set of self-conjugate partitions.
- The t-difference operator $D_t^{\mathcal{SC}}$ for self-conjugate partitions is defined by

$$D_t^{\mathcal{SC}} g(\lambda) := \sum_{\substack{\lambda^+ \in \mathcal{SC}, \lambda^+ \geq_t \lambda \\ |\lambda^+/\lambda| = 2t}} g(\lambda^+) - g(\lambda).$$

Theorem (X. 2015, joint with Han)

Let $t = 2t'$ be an even positive integer, μ be a given self-conjugate partition, and $u', v', j_u, j'_v, k_u, k'_v$ be nonnegative integers. Then we have

$$P(n) = (2t)^n n! \sum_{\substack{\lambda \in \mathcal{SC}, |\lambda| = 2nt \\ \#\mathcal{H}_t(\lambda) = 2n}} \frac{Q_1(h^2 : h \in \mathcal{H}(\lambda))\, Q_2(c : c \in \mathcal{C}(\lambda))}{H_t(\lambda)}$$

is a polynomial in n for any symmetric functions Q_1 and Q_2.

Corollary (Pétréolle 2015)

Let $t = 2t'$ be an even positive integer. Then

$$\sum_{\substack{\lambda \in \mathcal{SC}, |\lambda| = 2nt \\ \#\mathcal{H}_t(\lambda) = 2n}} \frac{1}{H_t(\lambda)} = \frac{1}{(2t)^n n!}.$$

Corollary

Let $t = 2t'$ be an even positive integer. We have

$$(2t)^n n! \sum_{\substack{\lambda \in \mathcal{SC}, |\lambda| = 2nt \\ \#\mathcal{H}_t(\lambda) = 2n}} \frac{1}{H_t(\lambda)} \sum_{h \in \mathcal{H}(\lambda)} h^2 = 6t^2 n^2 + \frac{1}{3}(t^2 - 6t - 1)tn,$$

$$(2t)^n n! \sum_{\substack{\lambda \in \mathcal{SC}, |\lambda| = 2nt \\ \#\mathcal{H}_t(\lambda) = 2n}} \frac{1}{H_t(\lambda)} \sum_{c \in \mathcal{C}(\lambda)} c^2 = 2t^2 n^2 + \frac{1}{3}(t^2 - 6t - 1)tn.$$

Doubled distinct partitions and strict partitions

- A strict partition (bar partition) is a finite strict decreasing sequence of positive integers $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_\ell)$.
- The doubled distinct partition $\psi(\bar{\lambda})$ of a strict partition $\bar{\lambda}$, is the usual partition whose Young diagram is obtained by adding $\bar{\lambda}_i$ boxes to the i-th column of the shifted Young diagram of $\bar{\lambda}$ for $1 \leq i \leq \ell(\bar{\lambda})$.
- For example, $(6, 4, 4, 1, 1)$ is the doubled distinct partition of $(5, 2, 1)$.

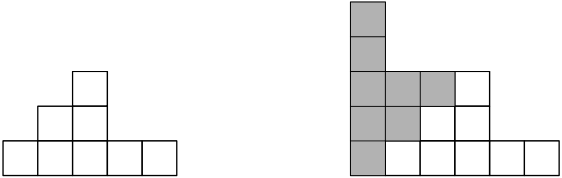

Figure: From strict partitions to doubled distinct partitions.

Doubled distinct partitions and strict partitions

- $\mathcal{DD}$: the set of doubled distinct partitions.
- The t-difference operator $D_t^{\mathcal{DD}}$ for doubled distinct partitions is defined by

$$D_t^{\mathcal{DD}} g(\lambda) = \sum_{\substack{\lambda^+ \in \mathcal{DD},\ \lambda^+ \geq_t \lambda \\ |\lambda^+/\lambda| = 2t}} g(\lambda^+) - g(\lambda).$$

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Theorem (X. 2015, joint with Han)

Let $t = 2t' + 1$ be an odd positive integer. The following summation for the positive integer n

$$(2t)^n n! \sum_{\substack{\lambda \in \mathcal{DD}, |\lambda| = 2nt \\ \#\mathcal{H}_t(\lambda) = 2n}} \frac{Q_1(h^2 : h \in \mathcal{H}(\lambda))\, Q_2(c : c \in \mathcal{C}(\lambda))}{H_t(\lambda)}$$

is a polynomial in n for any symmetric functions Q_1 and Q_2.

Doubled distinct partitions and strict partitions

Corollary (Pétréolle 2015)

Let $t = 2t' + 1$ be an odd positive integer. Then

$$\sum_{\substack{\lambda \in \mathcal{DD}, |\lambda| = 2nt \\ \#\mathcal{H}_t(\lambda) = 2n}} \frac{1}{H_t(\lambda)} = \frac{1}{(2t)^n n!}.$$

Corollary

Let $t = 2t' + 1$ be an odd positive integer. We have

$$(2t)^n n! \sum_{\substack{\lambda \in \mathcal{DD}, |\lambda| = 2nt \\ \#\mathcal{H}_t(\lambda) = 2n}} \frac{1}{H_t(\lambda)} \sum_{h \in \mathcal{H}(\lambda)} h^2 = 6t^2 n^2 + \frac{1}{3}(t^2 - 6t + 2)tn,$$

$$(2t)^n n! \sum_{\substack{\lambda \in \mathcal{DD}, |\lambda| = 2nt \\ \#\mathcal{H}_t(\lambda) = 2n}} \frac{1}{H_t(\lambda)} \sum_{c \in \mathcal{C}(\lambda)} c^2 = 2t^2 n^2 + \frac{1}{3}(t^2 - 6t + 2)tn.$$

Doubled distinct partitions and strict partitions

Corollary

Let Q be a given symmetric function, and $\bar{\mu}$ be a given strict partition. Then

$$P(n) = \sum_{|\bar{\lambda}/\bar{\mu}|=n} \frac{2^{|\bar{\lambda}|-|\bar{\mu}|-\ell(\bar{\lambda})+\ell(\bar{\mu})} \bar{f}_{\bar{\lambda}/\bar{\mu}}}{\bar{H}(\bar{\lambda})} Q\left(\binom{\bar{c}_\square}{2} : \square \in \bar{\lambda}\right)$$

is a polynomial of n. In particular,

$$\sum_{|\bar{\lambda}/\bar{\mu}|=n} \frac{2^{|\bar{\lambda}|-|\bar{\mu}|-\ell(\bar{\lambda})+\ell(\bar{\mu})} \bar{f}_{\bar{\lambda}/\bar{\mu}} \bar{H}(\bar{\mu})}{\bar{H}(\bar{\lambda})} \Big(\sum_{\square\in\bar{\lambda}} \binom{\bar{c}_\square}{2} - \sum_{\square\in\bar{\mu}} \binom{\bar{c}_\square}{2}\Big) = \binom{n}{2} + n|\bar{\mu}|.$$

Corollary

Suppose that k is a given nonnegative integer. Then

$$\sum_{|\bar{\lambda}|=n} \frac{2^{|\bar{\lambda}|-\ell(\bar{\lambda})} \bar{f}_{\bar{\lambda}}}{\bar{H}(\bar{\lambda})} \sum_{\square\in\bar{\lambda}} \binom{\bar{c}_\square + k - 1}{2k} = \frac{2^k}{(k+1)!} \binom{n}{k+1}.$$

Thank You for Listening!