A sextuple equidistribution arising in Pattern Avoidance

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Joint work with Dongsu Kim

Eulerian polynomials

Definition

The Eulerian polynomial $A_n(t)$ may be defined by Euler's basic formula (Leonhard Euler 1755):

$$\sum_{k\geq 0}(k+1)^n t^k = \frac{A_n(t)}{(1-t)^{n+1}}.$$

$$A_1(t) = 1$$
$$A_2(t) = 1 + t$$
$$A_3(t) = 1 + 4t + t^2$$
$$A_4(t) = 1 + 11t + 11t^2 + t^3$$
$$A_5(t) = 1 + 26t + 66t^2 + 26t^3 + t^4$$

Permutation Statistics

$\mathfrak{S}_n$: Set of permutations of $[n] := \{1, 2, \cdots, n\}$

Definition

For $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$:

$$\mathrm{DES}(\pi) := \{i \in [n-1] : \pi_i > \pi_{i+1}\}$$
$$\mathrm{des}(\pi) := |\mathrm{DES}(\pi)| \quad \text{(Descent number)}.$$

$$\mathrm{DES}(3.15.24) = \{1, 3\}$$

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Theorem (Riordan 1958)

$$A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)}.$$

Inversion sequences: $\mathfrak{I}_n = \{(e_1, e_2, \ldots, e_n) \in \mathbb{Z}^n : 0 \leq e_i < i\}$

$$\mathfrak{I}_3 = \{(0,0,0),(0,0,1),(0,0,2),(0,1,0),(0,1,1),(0,1,2)\}$$

Definition

For $e = (e_1, e_2, \cdots, e_n) \in \mathfrak{I}_n$:

$$\mathrm{ASC}(e) := \{i \in [n-1] : e_i < e_{i+1}\}$$
$$\mathsf{asc}(e) := |\mathrm{ASC}(e)| \quad \text{(Ascent number)}.$$

$$\mathrm{ASC}(0,1,1,2,0) = \{1,3\}$$

A natural bijection: inv-code

$$|\mathfrak{S}_n| = |\mathfrak{I}_n| = n!$$

and more...

$$\sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{des}(\pi)} = \sum_{e\in\mathfrak{I}_n} t^{\mathsf{asc}(e)}$$

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A natural bijection (inv-code) $\phi : \mathfrak{S}_n \to \mathfrak{I}_n$ with $\phi(\pi) = (e_1, \ldots, e_n)$, where

$$e_i = |\{j : j < i \text{ and } \pi_j > \pi_i\}|.$$

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$$e_i = |\{j : j < i \text{ and } \pi_j > \pi_i\}|.$$

This proves even more:

$$\sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{DES}(\pi)} = \sum_{e\in\mathfrak{I}_n} t^{\mathsf{ASC}(e)},$$

where $t^{\{i_1,\dots,i_k\}} := t_{i_1} \cdots t_{i_k}$.

Double Eulerian statistics

$\mathrm{dist}(e)$: number of distinct positive entries in e

Theorem (Dumont 1974)

$$\sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)} = \sum_{e \in \mathfrak{I}_n} t^{\mathrm{dist}(e)}.$$

Double Eulerian statistics

dist(e): number of distinct positive entries in e

Theorem (Dumont 1974)

$$\sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{des}(\pi)} = \sum_{e\in\mathfrak{I}_n} t^{\mathsf{dist}(e)}.$$

Via V-code and S-code:

Theorem (Foata 1977)

$$\sum_{\pi\in\mathfrak{S}_n} s^{\mathrm{des}(\pi^{-1})} t^{\mathrm{DES}(\pi)} = \sum_{e\in\mathfrak{I}_n} s^{\mathsf{dist}(e)} t^{\mathsf{ASC}(e)}.$$

- Rediscovered by Visontai (2013)
- An essentially different proof by Aas in PP 2013 (Paris)

Gessel's γ-positivity conjecture

Double Eulerian polynomials (Carlitz-Roselle-Scoville 1966):

$$A_n(s,t) := \sum_{\pi\in\mathfrak{S}_n} s^{\mathrm{des}(\pi^{-1})} t^{\mathrm{des}(\pi)}.$$

Conjectured by Gessel (2005):

Theorem (L. 2015)

The integers $\gamma_{n,i,j}$ are nonnegative in:

$$A_n(s,t) = \sum_{\substack{i,j\geq 0\\ j+2i\leq n-1}} \gamma_{n,i,j}(st)^i(1+st)^j(s+t)^{n-1-j-2i}.$$

Permutations without double descents

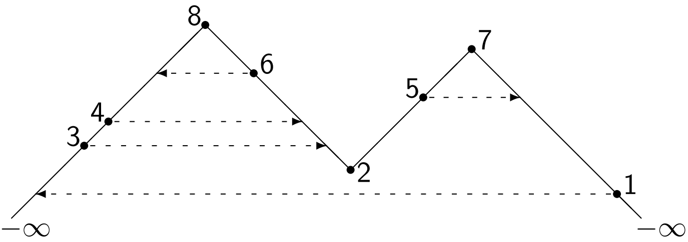

Figure : Foata-Strehl actions on 34862571

NDD_n: set of all permutations in $\mathfrak{S}_n$ without double descents

Theorem (Foata & Schützenberger 1970)

$$A_n(t) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,i} t^i (1+t)^{n+1-2i},$$

where $\gamma_{n,i} = \#\{\pi \in \mathrm{NDD}_n : \mathrm{des}(\pi) = i\}$.

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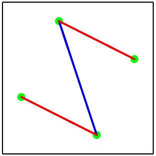
Problem

Is there any combinatorial interpretation for $\gamma_{n,i,j}$*?*

Separable permutations

Restrict to the terms without $s + t$:

$\pi = 2413$



$\mathrm{des}(\pi) = 1$
$\mathrm{des}(\pi^{-1}) = 2$
First $\mathrm{des}(\pi) \neq \mathrm{des}(\pi^{-1})$

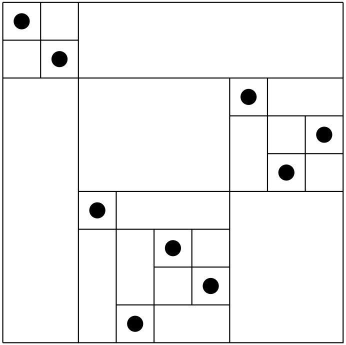
Definition

Permutations that avoid both the patterns 2413 and 3142 are separable permutations.

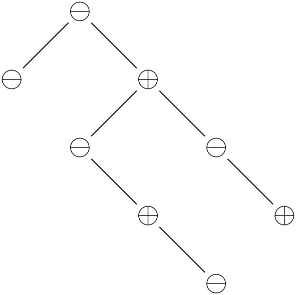
West (1995): $|\mathfrak{S}_n(2413, 3142)| = S_n$, the nth Large Schröder numbers.

Separable permutations

"di-sk" trees



bij. $\Longleftrightarrow$



Descent polynomial on Separable permutations

Via combinatorial approach using "di-sk" trees:

Theorem (Fu-L.-Zeng 2015)

$$\sum_{\pi\in\mathfrak{S}_n(2413,3142)} t^{\mathrm{des}(\pi)} = \sum_{k=0}^{\lfloor (n-1)/2\rfloor} \gamma_{n,k}^{S} t^k (1+t)^{n-1-2k},$$

where

$$\gamma_{n,k}^{S} = |\{\pi \in \mathfrak{S}_n(3142,2413)\cap \mathrm{NDD}_n : \mathrm{des}(\pi)=k\}|.$$

021-avoiding $\Leftrightarrow$ positive entries are weakly increasing

Via bijections with "di-sk" trees:

Theorem (Fu-L.-Zeng & Corteel et al. 2015)

$$\sum_{\pi\in\mathfrak{S}_n(2413,3142)} t^{\mathrm{des}(\pi)} = \sum_{e\in\mathfrak{I}_n(021)} t^{\mathrm{asc}(e)}.$$

Problem

$$\sum_{e\in\mathfrak{I}_n(021)} t^{\mathrm{asc}(e)} = \sum_{k=0}^{\lfloor (n-1)/2\rfloor} \gamma_{n,k}^{S} t^k (1+t)^{n-1-2k}$$

What is the combinatorial interpretation of $\gamma_{n,k}^{S}$ in terms of 021-avoiding inversion sequences?

Double Eulerian equidistribution

Theorem (Foata 1977)

$$\sum_{\pi\in\mathfrak{S}_n} s^{\mathrm{des}(\pi^{-1})} t^{\mathrm{DES}(\pi)} = \sum_{e\in\mathfrak{I}_n} s^{\mathsf{dist}(e)} t^{\mathsf{ASC}(e)}.$$

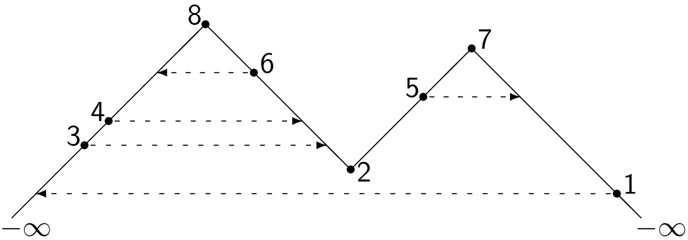
Restricted version of Foata's 1977 result:

Theorem (Kim-L. 2016)

$$\sum_{\pi\in\mathfrak{S}_n(2413,4213)} s^{\mathrm{des}(\pi^{-1})} t^{\mathrm{DES}(\pi)} = \sum_{e\in\mathfrak{I}_n(021)} s^{\mathsf{dist}(e)} t^{\mathsf{ASC}(e)}.$$

- Neither Foata's original bijection nor Aas' approach could be applied to prove this restricted version.

First application



As $\mathfrak{S}_n(2413, 4213)$ is invariant under Foata-Strehl action:

Corollary

$$\sum_{e \in \mathfrak{I}_n(021)} t^{\mathrm{asc}(e)} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k}^{S} t^k (1+t)^{n-1-2k},$$

where

$$\gamma_{n,k}^{S} = |\{e \in \mathfrak{I}_n(021) : e \textit{ has no double ascents}, \mathrm{asc}(e) = k\}|.$$

Second application

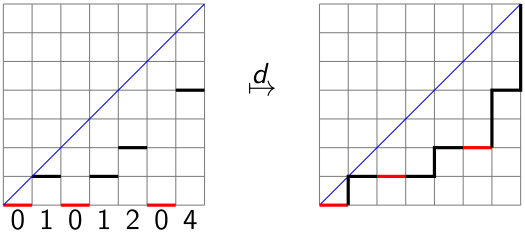


Figure : The outline of an inversion sequence

$$S = S(s,t;z) := \sum_{n\geq 1} z^n \sum_{\pi\in\mathfrak{S}_n(2413,4213)} s^{\mathrm{des}(\pi^{-1})} t^{\mathrm{des}(\pi)}$$

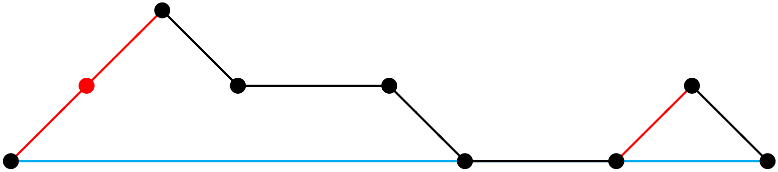
Theorem (Double Eulerian distribution)

$$S = t(z(s-1)+1)S + tz(2s-1)S^2 + z(ts+1)S + z.$$

Ascents on Schröder paths

A Schröder n-path is a lattice path on the plane from $(0,0)$ to $(2n,0)$, never going below x-axis, using the steps

$$(1,1) \quad (1,-1) \quad (2,0).$$



Corollary (Conjecture of Corteel et al. 2015)

An ascent in a Schröder path is a maximal string of consecutive up steps. Denoted by SP_n the set of Schröder n-path and by $\mathrm{asc}(p)$ the number of ascents of p. Then,

$$\sum_{e \in \mathfrak{I}_n(021)} s^{\mathrm{dist}(e)} = \sum_{p \in SP_{n-1}} s^{\mathrm{asc}(p)}.$$

A sextuple equidistribution (Statistics)

For each $\pi \in \mathfrak{S}_n$:

- $\mathrm{VID}(\pi) := \{2 \leq i \leq n : \pi_i$ appears to the right of $(\pi_i + 1)\}$, the **v**alues of **i**nverse **d**escents of π;
- $\mathrm{LMA}(\pi) := \{i \in [n] : \pi_i > \pi_j$ for all $1 \leq j < i\}$, the positions of **l**eft-to-right **ma**xima of π;
- $\mathrm{LMI}(\pi) := \{i \in [n] : \pi_i < \pi_j$ for all $1 \leq j < i\}$, the positions of **l**eft-to-right **mi**nima of π;
- $\mathrm{RMA}(\pi) := \{i \in [n] : \pi_i > \pi_j$ for all $j \geq i\}$, the positions of **r**ight-to-left **ma**xima of π;
- $\mathrm{RMI}(\pi) := \{i \in [n] : \pi_i < \pi_j$ for all $j \geq i\}$, the positions of **r**ight-to-left **mi**nima of π;

and for each $e \in \mathfrak{I}_n$:

- $\mathrm{DIST}(e) := \{2 \le i \le n : e_i \ne 0 \text{ and } e_i \ne e_j \text{ for all } j > i\}$, the positions of the last occurrence of **dist**inct positive entries of e;
- $\mathrm{ZERO}(e) := \{i \in [n] : e_i = 0\}$, the positions of **zero**s in e;
- $\mathrm{EMA}(e) := \{i \in [n] : e_i = i - 1\}$, the positions of the **e**ntries of e that achieve the **ma**ximum;
- $\mathrm{RMI}(e) := \{i \in [n] : e_i < e_j \text{ for all } j \ge i\}$, the positions of **r**ight-to-left **mi**nima of e.

A sextuple equidistribution (Main result)

Theorem (Kim-L. 2016)

There exists a bijection $\Psi : \mathfrak{I}_n(021) \to \mathfrak{S}_n(2413, 4213)$, *which transforms the sextuple*

$$(\mathrm{DIST}, \mathrm{ASC}, \mathrm{ZERO}, \mathrm{EMA}, \mathrm{RMI}, \mathrm{EXPO})$$

to

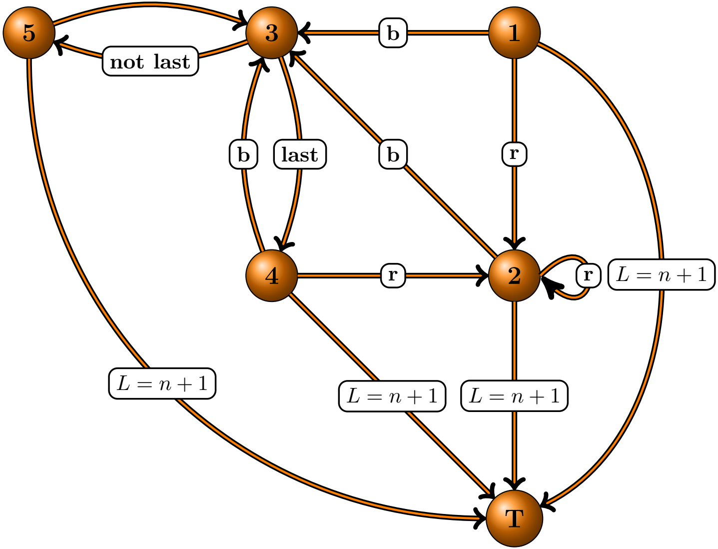
$$(\mathrm{VID}, \mathrm{DES}, \mathrm{LMA}, \mathrm{LMI}, \mathrm{RMA}, \mathrm{RMI}).$$

The algorithm Ψ

The labeling algorithm, where temporary variables L, H, P correspond to words *label, height, position*, works as follows:

1. (Start) $L \leftarrow 1$ (This means that 1 is assigned to L); draw the diagonal (line) $y = x$ on $d(e)$ and label the highest east step touched by the diagonal, say E_k, with L; $L \leftarrow L + 1$, $H \leftarrow d_k$, $P \leftarrow k$; go to (2), if E_P is a red east step (i.e. $k = 1$), otherwise go to (3);
2. draw the leftmost new line that touches at least one unlabeled black east step or a *labelable* red east step; label the highest east step touched by this new line, say E_k, with L; $L \leftarrow L + 1$, $H \leftarrow d_k$, $P \leftarrow k$; go to (2), if E_P is a red east step, otherwise go to (3);
3. go to (5), if there is a black east step E_j with $j > P$ and height $d_j = H$, otherwise go to (4);
4. move from E_P along the two-colored Dyck path $d(e)$ to the left and along the lines that were already drawn to the southwest until we arrive at the first unlabeled east step that is a black step or a *labelable* red step, say E_k; label E_k with L; $L \leftarrow L + 1$, $H \leftarrow d_k$, $P \leftarrow k$; go to (2), if E_P is a red east step, otherwise go to (3);
5. draw the leftmost line beginning at an east step right to E_P which touches at least one black east step; label the highest east step touched by this new line, say E_k, with L; $L \leftarrow L + 1$, $H \leftarrow d_k$, $P \leftarrow k$; go to (3).

Flowchart of Ψ



The algorithm Ψ (An example)

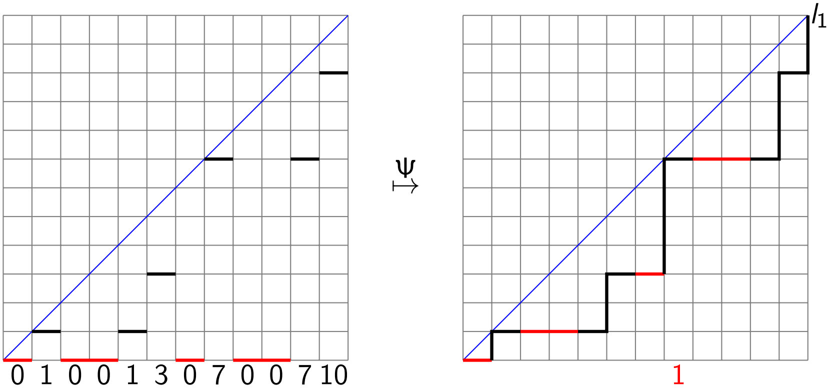


Figure : An example of the algorithm Ψ

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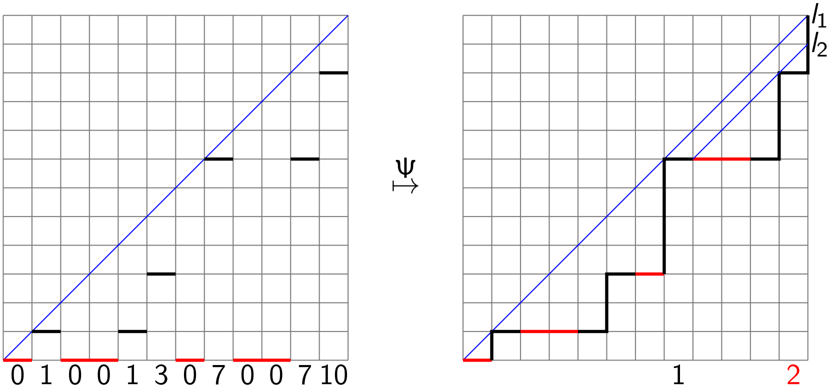

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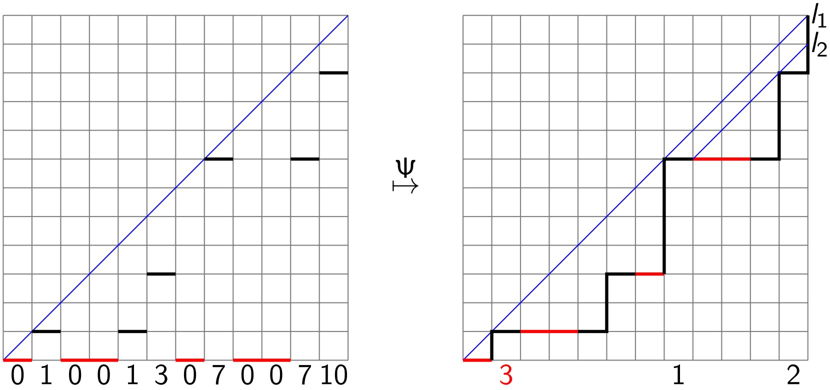

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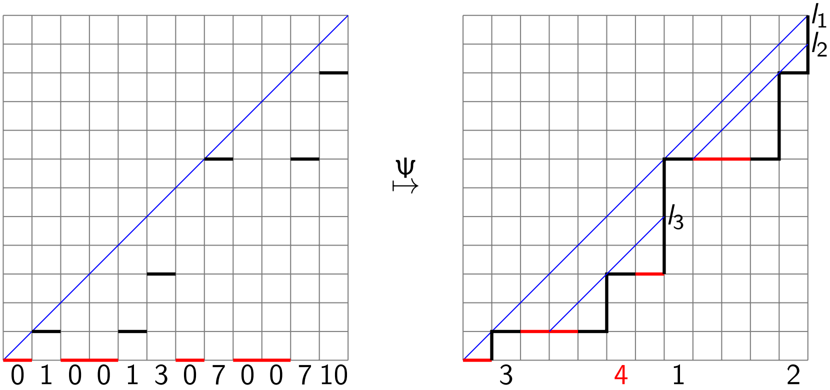

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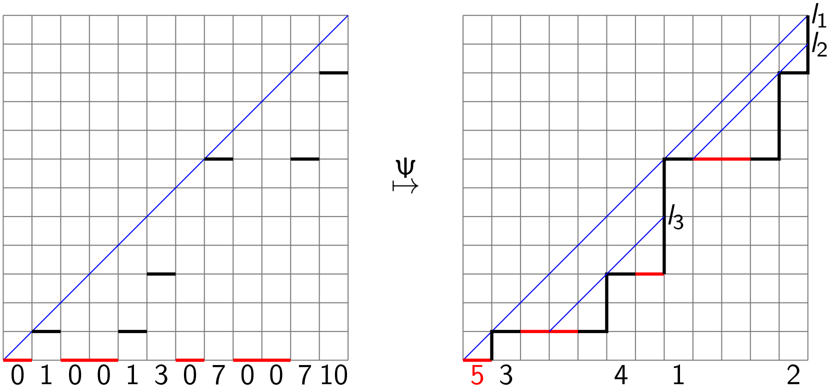

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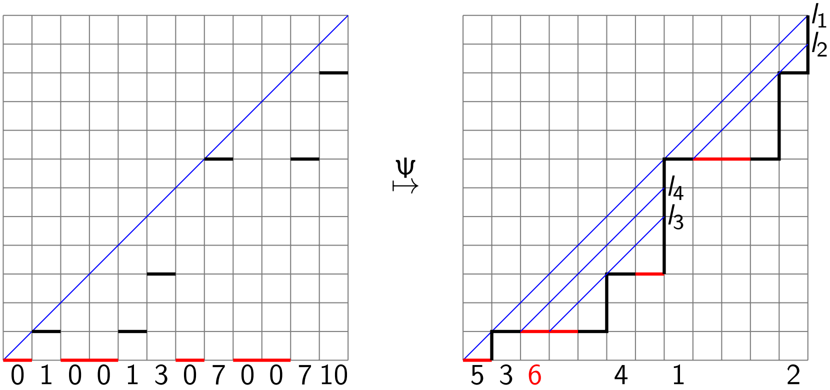

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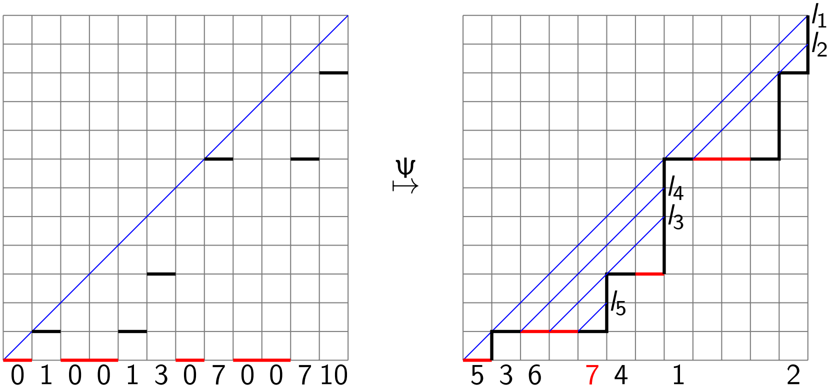

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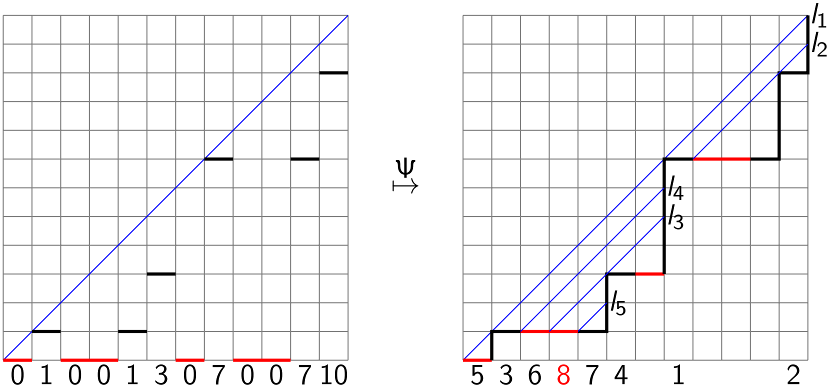

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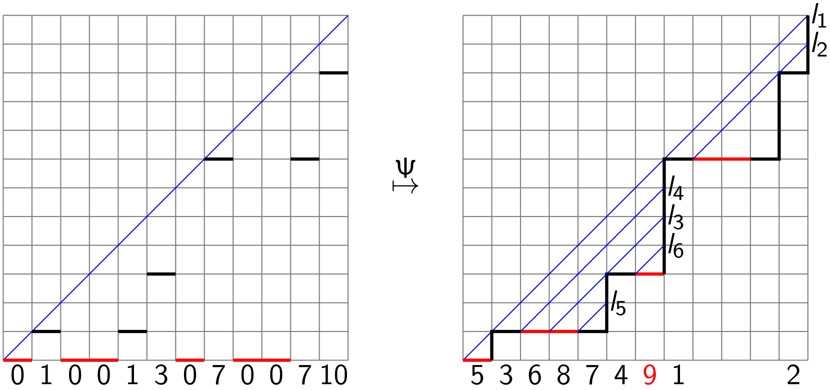

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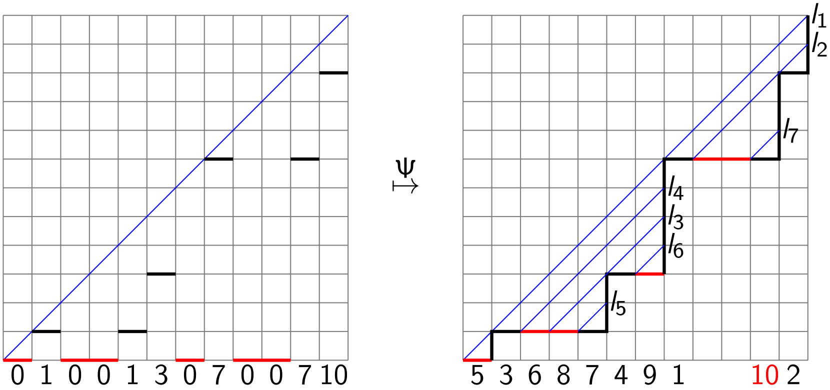

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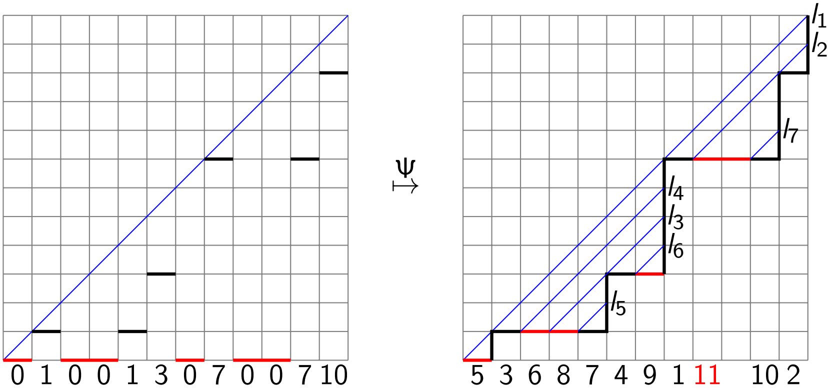

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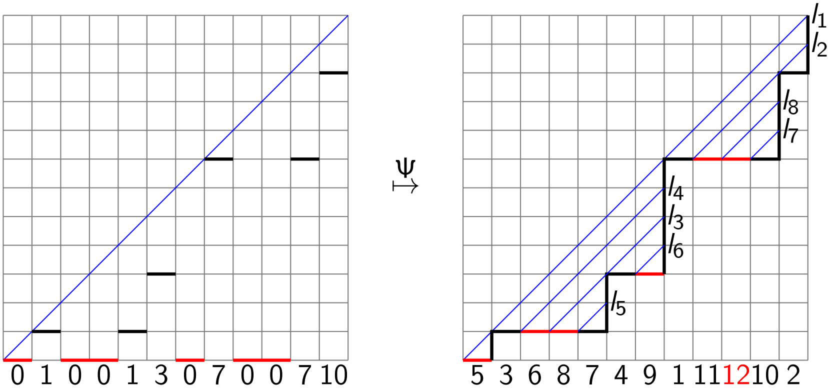

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Merci pour votre attention