Laplace Expansion of Schur Functions

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Séminaire Lotharingien de Combinatoire

Outline

1. Background and Notation
   - Sequences and Partitions
   - Schur Functions
2. Laplace Expansion
   - Concatenation of Partitions
   - Two Concatenation Identities for Schur Functions
3. Visual Interpretation of Concatenation
4. Application

Sequences

A sequence is a finite list of elements.

- length
- subsequence (not necessarily consecutive)
- addition (componentwise)
- union

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$\mathcal{S} = (5, 3)$
$\mathcal{T} = (4, 4, 0)$

$\mathcal{S} \cup \mathcal{T} = (5, 3, 4, 4, 0)$

Two Δ-Functions

Let $\mathcal{X} = (x_1, \ldots, x_n)$ and $\mathcal{Y} = (y_1, \ldots, y_m)$ be sequences.

- $\Delta(\mathcal{X}) = \prod_{1 \le i < j \le n} (x_i - x_j)$
- $\Delta(\mathcal{X}; \mathcal{Y}) = \prod_{1 \le i \le n} \prod_{1 \le j \le m} (x_i - y_j)$

Partitions

- A partition is a non-increasing sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of non-negative integers.
- The length of a partition is the number of its *positive* parts.
- We freely think of partitions as Young diagrams.

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- $\rho_n = (n-1, \dots, 1, 0)$
- $\langle m^n \rangle = \underbrace{(m, \dots, m)}_{n}$

Schur Functions

Definition

Let $\mathcal{X}$ be a set of variables of length n and λ a partition. If $l(\lambda) > n$, then $s_\lambda(\mathcal{X}) = 0$; otherwise,

$$s_\lambda(\mathcal{X}) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{1\leq i,j\leq n}}{\Delta(\mathcal{X})}.$$

The Schur function $s_\lambda(\mathcal{X})$ is a symmetric homogeneous polynomial of degree $|\lambda|$.

Laplace Expansion of Matrices

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix}$$

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Laplace Expansion of Matrices (formal statement)

Let A be an $n \times n$ matrix. For any subsequence $K \subset [n]$,

1. $$\det(A) = \sum_{\substack{J \subset [n]: \\ l(J) = l(K)}} \varepsilon(sort(K, J)) \det(A_{KJ}) \det\left(A_{[n]\setminus K\ [n]\setminus J}\right)$$

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Laplace Expansion of Schur Functions (Dehaye '12) (1/2)

$$\begin{pmatrix} x_1^{\lambda_1+5-1} & x_1^{\lambda_2+5-2} & x_1^{\lambda_3+5-3} & x_1^{\lambda_4+5-4} & x_1^{\lambda_5+5-5} \\ x_2^{\lambda_1+5-1} & x_2^{\lambda_2+5-2} & x_2^{\lambda_3+5-3} & x_2^{\lambda_4+5-4} & x_2^{\lambda_5+5-5} \\ x_3^{\lambda_1+5-1} & x_3^{\lambda_2+5-2} & x_3^{\lambda_3+5-3} & x_3^{\lambda_4+5-4} & x_3^{\lambda_5+5-5} \\ x_4^{\lambda_1+5-1} & x_4^{\lambda_2+5-2} & x_4^{\lambda_3+5-3} & x_4^{\lambda_4+5-4} & x_4^{\lambda_5+5-5} \\ x_5^{\lambda_1+5-1} & x_5^{\lambda_2+5-2} & x_5^{\lambda_3+5-3} & x_5^{\lambda_4+5-4} & x_5^{\lambda_5+5-5} \end{pmatrix}$$

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$$\lambda + \rho_5 \overset{sort}{=} (\mu + \rho_3) \cup (\nu + \rho_2)$$

Laplace Expansion of Schur Functions (Dehaye '12) (1/2)

$$\begin{pmatrix}
x_1^{\mu_1+3-1} & x_1^{\nu_1+2-1} & x_1^{\mu_2+3-2} & x_1^{\mu_3+3-3} & x_1^{\nu_2+2-2} \\
x_2^{\mu_1+3-1} & x_2^{\nu_1+2-1} & x_2^{\mu_2+3-2} & x_2^{\mu_3+3-3} & x_2^{\nu_2+2-2} \\
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$$s_\lambda(\mathcal{X}) = \sum_{\substack{\mathcal{S},\mathcal{T}\subset\mathcal{X}: \\ \mathcal{S}\cup_{3,2}\mathcal{T} \overset{sort}{=} \mathcal{X}}} \varepsilon(sort) s_\mu(\mathcal{S}) s_\nu(\mathcal{T}) \frac{\Delta(\mathcal{S})\Delta(\mathcal{T})}{\Delta(\mathcal{X})}$$

Laplace Expansion of Schur Functions (Dehaye '12) (2/2)

$$\lambda + \rho_5 \overset{sort}{=} (\mu + \rho_3) \cup (\nu + \rho_2)$$

$$s_\lambda(\mathcal{X}) = \sum_{\substack{\mathcal{S},\mathcal{T}\subset\mathcal{X}: \\ \mathcal{S}\cup_{3,2}\mathcal{T}\overset{sort}{=}\mathcal{X}}} \varepsilon(sort) s_\mu(\mathcal{S}) s_\nu(\mathcal{T}) \frac{\Delta(\mathcal{S})\Delta(\mathcal{T})}{\Delta(\mathcal{X})}$$

$$= \sum_{\substack{\mathcal{S},\mathcal{T}\subset\mathcal{X}: \\ \mathcal{S}\cup_{3,2}\mathcal{T}\overset{sort}{=}\mathcal{X}}} \frac{\varepsilon(sort) s_\mu(\mathcal{S}) s_\nu(\mathcal{T})}{\Delta(\mathcal{S};\mathcal{T})}$$

Concatenation of Partitions

Definition

Let μ and ν be two partitions of length at most m and n, respectively. The (m,n)-concatenation of μ and ν, denoted $\mu \star_{m,n} \nu$, is the partition that satisfies

$$\mu \star_{m,n} \nu + \rho_{m+n} \overset{sort}{=} (\mu + \rho_m) \cup (\nu + \rho_n)$$

if it exists; otherwise, we set $\mu \star_{m,n} \nu = \infty$. Here, ∞ is just a symbol with the property that $s_\infty(\mathcal{X}) = 0$ for any set of variables $\mathcal{X}$.
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$$(5,1) \star_{2,4} (3,3) = \infty$$

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$$(5,1) \star_{3,2} (3,3) = (7,4,3,2,0) - \rho_5 = (3,1,1,1,0)$$

First Concatenation Identity for Schur Functions

Lemma (Dehaye '12)

Let the set $\mathcal{X}$ consist of $m+n$ variables. For any pair of partitions μ and ν with at most m and n parts, respectively, it holds that

$$s_{\mu \star_{m,n} \nu}(\mathcal{X}) = \sum_{\substack{\mathcal{S},\mathcal{T} \subset \mathcal{X}: \\ \mathcal{S} \cup_{m,n} \mathcal{T} \overset{sort}{=} \mathcal{X}}} \frac{\varepsilon(\mu,\nu) s_\mu(\mathcal{S}) s_\nu(\mathcal{T})}{\Delta(\mathcal{S};\mathcal{T})}.$$

Laplace Expansion of Schur Functions (Transposed)

$$\begin{pmatrix} x_1^{\lambda_1+5-1} & x_1^{\lambda_2+5-2} & x_1^{\lambda_3+5-3} & x_1^{\lambda_4+5-4} & x_1^{\lambda_5+5-5} \\ x_2^{\lambda_1+5-1} & x_2^{\lambda_2+5-2} & x_2^{\lambda_3+5-3} & x_2^{\lambda_4+5-4} & x_2^{\lambda_5+5-5} \\ x_3^{\lambda_1+5-1} & x_3^{\lambda_2+5-2} & x_3^{\lambda_3+5-3} & x_3^{\lambda_4+5-4} & x_3^{\lambda_5+5-5} \\ x_4^{\lambda_1+5-1} & x_4^{\lambda_2+5-2} & x_4^{\lambda_3+5-3} & x_4^{\lambda_4+5-4} & x_4^{\lambda_5+5-5} \\ x_5^{\lambda_1+5-1} & x_5^{\lambda_2+5-2} & x_5^{\lambda_3+5-3} & x_5^{\lambda_4+5-4} & x_5^{\lambda_5+5-5} \end{pmatrix}$$

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Second Concatenation Identity for Schur Functions

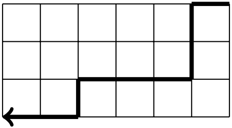
Lemma (HR '16)

Let $\mathcal{S}$ and $\mathcal{T}$ be sets consisting of m and n variables, respectively. For any partition λ, it holds that

$$s_\lambda(\mathcal{S} \cup \mathcal{T}) = \sum_{\substack{\mu,\nu: \\ \mu \star_{m,n} \nu = \lambda}} \frac{\varepsilon(\mu,\nu) s_\mu(\mathcal{S}) s_\nu(\mathcal{T})}{\Delta(\mathcal{S};\mathcal{T})}.$$

Staircase Walks

Let $\mathfrak{P}(m, n)$ be the set of all staircase walks going from the top-right to the bottom-left of an $m \times n$ rectangle.



This is an example of a staircase walk $\pi \in \mathfrak{P}(3, 6)$.

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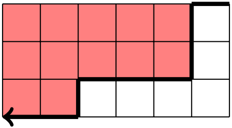

To $\pi \in \mathfrak{P}(3, 6)$, we associate the partition $\mu_\pi = (5, 5, 2) \subset \langle 6^3 \rangle$.

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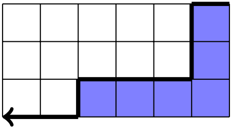

To $\pi \in \mathfrak{P}(3, 6)$, we also associate the partition $\nu_\pi = (4, 1, 1) \subset \langle 6^3 \rangle$.

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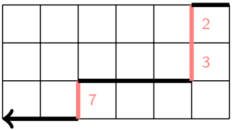

To $\pi \in \mathfrak{P}(3, 6)$, we associate the sequences
$v(\pi) = (2, 3, 7) \subset [9]$ and $h(\pi) = (1, 4, 5, 6, 8, 9) \subset [9]$.

How the attributes of π interact

Remark

Let $\pi \in \mathfrak{P}(m, n)$.

- For $i \in [m]$, $(\mu_\pi)_i + m - i = m + n - v(\pi)_i$.
- For $j \in [n]$, $\left(\nu'_\pi\right)_j + n - j = m + n - h(\pi)_j$.

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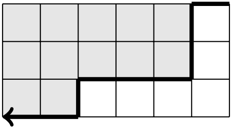

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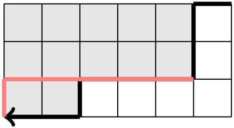

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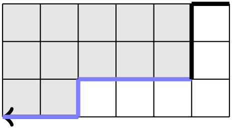


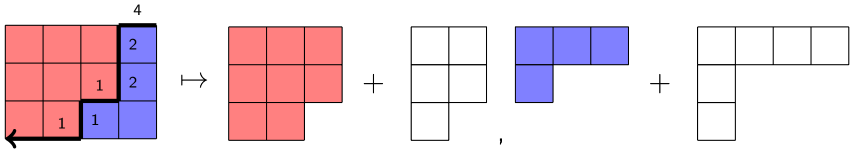
Labeled Staircase Walks and Concatenation

Lemma (HR '16)

For a fixed partition λ of length at most $m+n$, there is a 1-to-1 correspondence between $\mathfrak{P}(m,n)$ and $\{(\mu,\nu) : \mu \star_{m,n} \nu = \lambda\}$ given by

$$\pi \mapsto \left(\mu_\pi + \lambda_{v(\pi)}, \nu'_\pi + \lambda_{h(\pi)}\right).$$

Moreover, $\varepsilon\left(\mu_\pi + \lambda_{v(\pi)}, \nu'_\pi + \lambda_{h(\pi)}\right) = (-1)^{|\nu_\pi|}$.



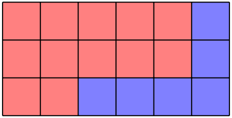
Complement of a Partition

Skip

Definition

The (m, n)-complement of a partition λ contained in the rectangle $\langle m^n \rangle$ is given by

$$\tilde{\lambda} = (m - \lambda_n, \ldots, m - \lambda_1) \subset \langle m^n \rangle.$$



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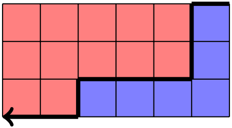

Complement of a Partition

Definition

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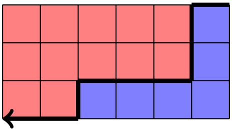

$$s_\lambda(\mathcal{X}) = \prod_{x \in \mathcal{X}} x^m s_{\tilde{\lambda}}(\mathcal{X}^{-1})$$

New Proof for the Dual Cauchy Identity

Dual Cauchy Identity

Let $\mathcal{X}$ and $\mathcal{Y}$ be two sets of variables, then

$$\sum_{\lambda} s_{\lambda}(\mathcal{X}) s_{\lambda'}(\mathcal{Y}) = \prod_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} (1 + xy).$$

New Proof for the Dual Cauchy Identity

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$$\begin{aligned}
lhs &= \sum_{\lambda \subset \langle m^n \rangle} \prod_{x \in \mathcal{X}} x^m s_{\tilde{\lambda}} \left(\mathcal{X}^{-1}\right) s_{\lambda'}(\mathcal{Y}) \\
&= \prod_{x \in \mathcal{X}} x^m \sum_{\pi \in \mathfrak{P}(n,m)} (-1)^{|\nu_\pi|} s_{\mu_\pi}(\mathcal{X}^{-1}) s_{\nu'_\pi}(-\mathcal{Y}) \\
&= \prod_{x \in \mathcal{X}} x^m \Delta \left(\mathcal{X}^{-1}, -\mathcal{Y}\right) = rhs
\end{aligned}$$

Littlewood-Schur Functions

Definition

Let $\mathcal{X}$ and $\mathcal{Y}$ be two sets of variables. Define

$$LS_\lambda(\mathcal{X};\mathcal{Y}) = \sum_{\mu,\nu} c_{\mu\nu}^{\lambda} s_\mu(\mathcal{X}) s_{\nu'}(\mathcal{Y})$$

where $c_{\mu\nu}^{\lambda}$ are Littlewood-Richardson coefficients.

Littlewood-Schur Functions

Definition

Let $\mathcal{X}$ and $\mathcal{Y}$ be two sets of variables. Define

$$LS_\lambda(\mathcal{X};\mathcal{Y}) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(\mathcal{X}) s_{\nu'}(\mathcal{Y})$$

where $c^\lambda_{\mu\nu}$ are Littlewood-Richardson coefficients.

The Littlewood-Schur function $LS_\lambda(\mathcal{X};\mathcal{Y})$ is a homogeneous polynomial of degree $|\lambda|$ which is symmetric in both $\mathcal{X}$ and $\mathcal{Y}$.

Application: Ratios Theorem

Let $U(N)$ be the group of unitary matrices of size N
endowed with its unique Haar measure of volume 1,
and let χ_g stand for the characteristic polynomial of the matrix $g \in U(N)$.

Average of Ratios of Characteristic Polynomials

$$\int_{U(N)} \frac{\prod_{\alpha \in \mathcal{A}} \chi_g(\alpha) \prod_{\beta \in \mathcal{B}} \chi_{g^{-1}}(\beta)}{\prod_{\delta \in \mathcal{D}} \chi_g(\delta) \prod_{\gamma \in \mathcal{C}} \chi_{g^{-1}}(\gamma)} dg$$

Thank you for your attention!

Questions

- Littlewood-Schur functions: other names
- Littlewood-Schur functions: determinantal formula
- index of a partition
- Littlewood-Schur functions: concatenation identities
- link to Number Theory
- Ratios Theorem: new expression
- Ratios Theorem: proof

Other Names for Littlewood-Schur Functions

▸ more questions?

Littlewood-Schur functions are also called:

- hook Schur functions (Berele and Regev 1987)
- supersymmetric polynomials (Nicoletti et al. 1981)
- super-Schur functions (Brenti 1993)
- Macdonald denotes them $s_\lambda(x/y)$

Determinantal Formula for Littlewood-Schur Functions

more questions?

Theorem (Moens and van der Jeugt '02)

Let $\mathcal{X}$ and $\mathcal{Y}$ be sets of variables with n and m elements, respectively, and let λ be a partition with (m, n)-index k. If k is negative, then $LS_\lambda(-\mathcal{X}; \mathcal{Y}) = 0$; otherwise,

$$LS_\lambda(-\mathcal{X}; \mathcal{Y}) = \varepsilon(\lambda) \frac{\Delta(\mathcal{Y}; \mathcal{X})}{\Delta(\mathcal{X})\Delta(\mathcal{Y})} \det \begin{pmatrix} \left((x-y)^{-1}\right)_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} & \left(x^{\lambda_j + n - m - j}\right)_{\substack{x \in \mathcal{X} \\ 1 \leq j \leq n-k}} \\ \left(y^{\lambda'_i + m - n - i}\right)_{\substack{1 \leq i \leq m-k \\ y \in \mathcal{Y}}} & 0 \end{pmatrix}$$

where $\varepsilon(\lambda) = (-1)^{\left|\lambda_{[n-k]}\right|} (-1)^{mk} (-1)^{k(k-1)/2}$.

Index of a Partition

▸ more questions?

Definition

The (m, n)-index of a partition λ is the largest integer k with the properties that $(m+1-k, n+1-k) \notin \lambda$ and $k \leq \min\{m, n\}$.

Index of a Partition

▸ more questions?

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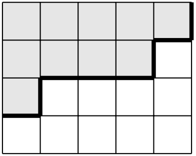

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▸ more questions?

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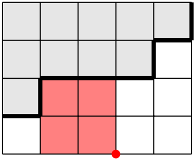
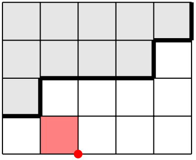


The $(3, 4)$-index of $\lambda = (5, 4, 1)$ is 2.

▸ more questions?

Definition

The (m, n)-index of a partition λ is the largest integer k with the properties that $(m + 1 - k, n + 1 - k) \notin \lambda$ and $k \leq \min\{m, n\}$.



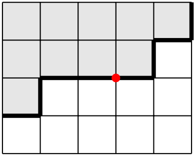
The $(2, 4)$-index of $\lambda = (5, 4, 1)$ is 1.

Index of a Partition

▸ more questions?

Definition

The (m, n)-index of a partition λ is the largest integer k with the properties that $(m + 1 - k, n + 1 - k) \notin \lambda$ and $k \leq \min\{m, n\}$.



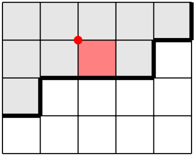
The $(3, 2)$-index of $\lambda = (5, 4, 1)$ is 0.

Index of a Partition

▸ more questions?

Definition

The (m, n)-index of a partition λ is the largest integer k with the properties that $(m+1-k, n+1-k) \notin \lambda$ and $k \leq \min\{m, n\}$.



The $(2, 1)$-index of $\lambda = (5, 4, 1)$ is -1.

Concatenation Identities for Littlewood-Schur Functions

more questions?

Proposition (HR '16)

Let $\mathcal{X}$ and $\mathcal{Y}$ be sets of variables with n and m elements, respectively and let the partition λ have (m,n)-index k. If $\lambda_{[n-k]} = \mu \star_{l,n-k-l} \nu$ for some integer $0 \leq l \leq \min\{n-k, n\}$, then $LS_\lambda(-\mathcal{X};\mathcal{Y}) =$

$$\sum_{\substack{\mathcal{S},\mathcal{T}\subset\mathcal{X}: \\ \mathcal{S}\cup_{l,n-l}\mathcal{T}\overset{sort}{=}\mathcal{X}}} \frac{\varepsilon(\mu,\nu) LS_{\mu+\langle k^l\rangle}(-\mathcal{S};\mathcal{Y}) LS_{\nu\cup\lambda_{(n+1-k,n+2-k,\dots)}}(-\mathcal{T};\mathcal{Y})}{\Delta(\mathcal{T};\mathcal{S})}.$$

Concatenation Identities for Littlewood-Schur Functions

more questions?

Proposition (HR '16)

Let $0 \leq l \leq \min\{n-k, n\}$. Let $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{Y}$ be sets containing l, $n-l$ and m variables, respectively. Suppose that k is the (m,n)-index of a partition λ, then $LS_\lambda(-(\mathcal{S}\cup\mathcal{T});\mathcal{Y}) =$

$$\sum_{p=0}^{\min\{l,m\}} \sum_{\substack{\mathcal{U},\mathcal{V}\subset\mathcal{Y}: \\ \mathcal{U}\cup_{p,m-p}\mathcal{V} \overset{sort}{=} \mathcal{Y}}} \sum_{\substack{\mu,\nu: \\ \mu\star_{l-p,n-k-l+p}\nu=\lambda_{[n-k]}}} \frac{\Delta(\mathcal{V};\mathcal{S})\Delta(\mathcal{T};\mathcal{U})}{\Delta(\mathcal{V};\mathcal{U})\Delta(\mathcal{T};\mathcal{S})}$$

$$\times\varepsilon(\mu,\nu) LS_{\mu-\langle (m-k)^{l-p}\rangle}(-\mathcal{S};\mathcal{U}) LS_{\nu\cup\lambda_{(n+1-k,n+2-k,\dots)}}(-\mathcal{T};\mathcal{V}).$$

Link to Number Theory

▸ more questions?

Conjecture

Some families of L-functions behave like the family of characteristic polynomials of unitary matrices.

$$\frac{1}{T}\int_0^T \frac{\prod_{\alpha\in\mathcal{A}}\zeta\left(1/2+it+\alpha\right)\prod_{\beta\in\mathcal{B}}\zeta\left(1/2-it+\beta\right)}{\prod_{\delta\in\mathcal{D}}\zeta\left(1/2+it+\delta\right)\prod_{\gamma\in\mathcal{C}}\zeta\left(1/2-it+\gamma\right)}dt$$

$$\sim$$

$$\int_{U(N)} \frac{\prod_{\alpha\in\mathcal{A}}\chi_g\left(e^{-\alpha}\right)\prod_{\beta\in\mathcal{B}}\chi_{g^{-1}}\left(e^{-\beta}\right)}{\prod_{\delta\in\mathcal{D}}\chi_g\left(e^{-\delta}\right)\prod_{\gamma\in\mathcal{C}}\chi_{g^{-1}}\left(e^{-\gamma}\right)}dg$$

as $N=\log T/2\pi$ goes to ∞.

Ratios Theorem

▸ more questions?

Ratios Theorem (HR '16)

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be sets of variables with elements in $\mathbb{C} \setminus \{0\}$. Suppose that all elements of $\mathcal{C} \cup \mathcal{D}$ have absolute value strictly less than 1 and that $l(\mathcal{C} \cup \mathcal{D}) \leq N$. If the elements of $\mathcal{A} \cup \mathcal{B}^{-1}$ are pairwise distinct, then

$$\int_{U(N)} \frac{\prod_{\alpha \in \mathcal{A}} \chi_g(\alpha) \prod_{\beta \in \mathcal{B}} \chi_{g^{-1}}(\beta)}{\prod_{\delta \in \mathcal{D}} \chi_g(\delta) \prod_{\gamma \in \mathcal{C}} \chi_{g^{-1}}(\gamma)} dg$$

$$= \frac{e_{l(\mathcal{B})}^{-l(\mathcal{A})}(\mathcal{B}) e_{l(\mathcal{D})}^{l(\mathcal{B})-l(\mathcal{C})}(\mathcal{D}) e_{l(\mathcal{C})}^{l(\mathcal{A})}(\mathcal{C})}{\Delta(\mathcal{A}; \mathcal{B}^{-1}) \Delta(\mathcal{D}^{-1}; \mathcal{C})} \sum_{k=0}^{\min\{l(\mathcal{A}), l(\mathcal{B})\}} (-1)^k e_{l(\mathcal{C})}^{-k}(\mathcal{C}) e_{l(\mathcal{D})}^{-k}(\mathcal{D})$$

$$\times \sum_{\substack{\mathcal{S}, \mathcal{T} \subset \mathcal{A}: \\ \mathcal{S} \cup_{k, l(\mathcal{A})-k} \mathcal{T} \overset{sort}{=} \mathcal{A}}} \frac{e_k^{N-l(\mathcal{D})+l(\mathcal{A})+l(\mathcal{B})-k}(\mathcal{S}) \Delta(\mathcal{C}^{-1}; \mathcal{T}) \Delta(\mathcal{S}; \mathcal{D})}{\Delta(\mathcal{T}; \mathcal{S})}$$

$$\times \sum_{\substack{\mathcal{X}, \mathcal{Y} \subset \mathcal{B}: \\ \mathcal{X} \cup_{k, l(\mathcal{B})-k} \mathcal{Y} \overset{sort}{=} \mathcal{B}}} \frac{e_k^{N-l(\mathcal{C})+l(\mathcal{A})+l(\mathcal{B})-k}(\mathcal{X}) \Delta(\mathcal{D}^{-1}; \mathcal{Y}) \Delta(\mathcal{X}; \mathcal{C})}{\Delta(\mathcal{Y}; \mathcal{X})}$$

$$\times \Delta(\mathcal{T}; \mathcal{X}^{-1}) \Delta(\mathcal{Y}; \mathcal{S}^{-1}).$$

Ratios Theorem: First Lines of the Proof

more questions?

$$\int_{U(N)} \det(g)^{l(\mathcal{B})} \prod_{\substack{x\in\mathcal{A}\cup\mathcal{B}^{-1}\\ \rho\in\mathcal{R}(g)}} (1 - x\overline{\rho}) \prod_{\substack{\delta\in\mathcal{D}\\ \rho\in\mathcal{R}(g)}} (1 - \delta\overline{\rho})^{-1} \prod_{\substack{\gamma\in\mathcal{C}\\ \rho\in\mathcal{R}(g)}} (1 - \gamma\rho)^{-1} dg$$

$$=$$

$$\int_{U(N)} e_N^{l(\mathcal{B})}(\mathcal{R}(g)) \sum_{\lambda} LS_{\lambda'}(-\left(\mathcal{A}\cup\mathcal{B}^{-1}\right); \mathcal{D})\overline{s_\lambda(\mathcal{R}(g))} \sum_{\kappa} s_\kappa(\mathcal{C}) s_\kappa(\mathcal{R}(g)) dg$$

$$=$$

$$\sum_{\lambda} LS_{(\lambda+\langle l(\mathcal{B})^N\rangle)'}(-\left(\mathcal{A}\cup\mathcal{B}^{-1}\right); \mathcal{D}) s_\lambda(\mathcal{C})$$